CMPT 210: Probability and Computing

Lecture 4

Sharan Vaswani

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Recap

Sum rule: If $A_1, A_2 \dots A_m$ are disjoint sets, then, $|A_1 \cup A_2 \cup \dots \cup A_m| = \sum_{i=1}^{m} |A_i|$ (E.g Number of rainy, snowy or hot days in the year).

Product Rule: For sets A_1, $A_2 \dots, A_m$, $|A_1 \times A_2 \times \dots \times A_m| = \prod_{i=1}^{m} |A_i|$ (E.g: Selecting one course each from every subject)

Generalized product rule: If S is the set of length k sequences such that the first entry can be selected in n_1 ways, after the first entry is chosen, the second one can be chosen in n_2 ways, and so on, then $|S| = n_1 \times n_2 \times \dots n_k$. (E.g Number of ways n people can be arranged in a line $= n!$)

Division rule: $f : A \to B$ is a k-to-1 function, then, $|A| = k|B|$. (E.g. For arranging people around a round table, f : seatings $\to$ arrangements is an n-to-1 function).

Number of ways of choosing size k-subsets from a size n-set: $\binom{n}{k}$ (E.g. Number of n-bit sequences with exactly k ones).

Binomial Theorem: For all $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$, $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$.

Counting Practice

Q: Prove Pascal's identity using a combinatorial proof: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Consider n students in this class. What is the number of ways of selecting k students? $\binom{n}{k}$.

What is the number of ways of selecting k students if we have to ensure to include a particular student? $\binom{n-1}{k-1}$.

What is the number of ways of selecting k students if we have to ensure to NOT include a particular student? $\binom{n-1}{k}$.

Number of ways to select k students = number of ways of selecting k students to include a particular student + number of ways of selecting k students to NOT include a particular student. Hence, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Counting Practice

Q: A standard dice (with numbers $\{1, 2, 3, 4, 5, 6\}$) is rolled 6 times in succession. Define a roll to be the sequence formed by the numbers on the 6 dice. For example, (1,2,4,1,6,5).

- How many rolls will have no 6? Ans: 5^6. Since each of the 6 positions (corresponding to a roll) can be filled in 5 ways (all except 6).
- How many rolls will have each number exactly once? Ans: $6!$. The first position can be filled in 6 ways. Since we need each number to come up only once, the second position can be filled in 5 ways, and so on.
- How many rolls will have 6 come up exactly once? Ans: In order to count such rolls, first pick the roll number where the 6 appears. This can be done in 6 ways. For each choice, the remaining 5 positions (corresponding to the rolls) can be filled in 5 ways. Hence, the total number of ways $= 6 \times 5^5$
- How many rolls will have 6 come up exactly k times (for $k \leq 6$)? Ans: Similar to the above question, first pick the k rolls where a 6 appears, and the remaining $6 - k$ positions can be filled in 5 ways. Hence, the total number of ways $= \binom{6}{k} \times 5^{6-k}$

Counting Practice

Q: How many 5 digit numbers are there which contain at least one zero? Note that a number is different from a string, i.e. 01234 is not a 5-digit number and is hence not allowed.

Ans: We need to first choose the number of zeros and then fill out the remaining digits. The number of zeros can be $\{1, 2, 3, 4\}$ (since 00000 is not a 5 digit number). Let D_i be the set of numbers that contain i zeros. Hence, the total number of numbers is $|D_1 \cup D_2 \cup D_3 \cup D_4| = \sum_{i=1}^{4} |D_i|$ (using the sum rule since the sets are disjoint). $|D_i| = \binom{4}{i} \times 9^{5-i}$ corresponding to choosing i positions for the zero (the first position cannot be chosen) and filling the remaining positions by one of the remaining 9 digits. Hence, the total number of ways is equal to $\sum_{i=1}^{4} \binom{4}{i} \times 9^{5-i}$.

Ans: **Alternative way**: Define A be the total number of 5 digit numbers, and Z be the number of 5 digit numbers with no zeros. If D is the number of 5 digit numbers with at least one zero. Then, $D \cup Z = A$. Since D and Z are disjoint, by the sum rule, $|D| = |A| - |Z|$. $|A| = 9 \times 10^4$ (the first digit cannot be zero). $|Z| = 9^5$ (no digit can be zero). Hence, $|D| = 9 \times 10^4 - 9^5$.

Counting Practice

Q: How many non-negative integer solutions ($x_1, x_2, x_3 \geq 0$) are there to the following equation:

$$x_1 + x_2 + x_3 = 40$$

Ans: This is similar to the donuts question where there are 3 donuts varieties – x_1, x_2 and x_3, and we need to choose 40 donuts. Using the earlier results, the number of possible ways this can be done is equal to the number of strings with length 42 with exactly 2 ones, which is equal to $\binom{42}{2}$.

Questions?

Generalization to Multinomials

We saw how to split a set into two subsets - one that contains some elements, while the other does not. Can generalize the arguments to split a set into more than two subsets.

Definition: A $(k_1, k_2, \ldots, k_m)$-split of set A is a sequence of sets $(A_1, A_2, \ldots A_m)$ s.t. sets A_i form a partition ($A_1 \cup A_2 \cup \ldots = A$ and for $i \neq j$, $A_i \cap A_j = \emptyset$) and $|A_i| = k_i$.

Example: A $(2, 1, 3)$-split of $A = \{1, 2, 3, 4, 5, 6\}$ is $(\{2, 4\}, \{1\}, \{3, 5, 6\})$. Here, $m = 3$, $A_1 = \{2, 4\}$, $A_2 = \{1\}$, $A_3 = \{3, 5, 6\}$ s.t. $|A_1| = 2$, $|A_2| = 1$, $|A_3| = 3$, $A_1 \cup A_2 \cup A_3 = A$ and for $i \neq j$, $A_i \cap A_j = \emptyset$.

Example: Consider strings of length 6 of a's, b's and c's such that number of a's $= 2$; number of b's $= 1$ and number of c's $= 3$. Possible strings: abaccc, ccbaac, bacacc, cbacac.

Each possible string, e.g. bacacc can be written as a $(2, 1, 3)$-split of $A = \{1, 2, 3, 4, 5, 6\}$ as $(\{2, 4\}, \{1\}, \{3, 5, 6\})$ where A_1 records the positions of a, A_2 records the positions of b and A_3 records the positions of c.

Generalization to Multinomials

Q: Show that the number of ways to obtain an $(k_1, k_2, \ldots, k_m)$ split of A with $|A| = n$ is $\binom{n}{k_1, k_2, \ldots k_m} = \frac{n!}{k_1!\, k_2!\, \ldots k_m!}$ where $\sum_i k_i = n$.

Can map any permutation $(a_1, a_2, \ldots a_n)$ into a split by selecting the first k_1 elements to form set A_1, next k_2 to form set A_2 and so on. For the same split, the order of the elements in each subset does not matter. Hence f : number of permutations $\to$ number of splits is a $k_1!\, k_2!\, \ldots k_m!$-to-1 function.

Hence, $|\text{number of splits}| = \frac{|\text{number of permutations}|}{k_1!\, k_2!\, \ldots k_m!} = \frac{n!}{k_1!\, k_2!\, \ldots k_m!}$.

Alternative way: Form the sets A_1, $A_2 \ldots A_m$ sequentially. Number of ways to choose k_1 elements to form $A_1 = \binom{n}{k_1}$. After forming A_1, we have $n - k_1$ elements left. From these remaining elements, number of ways to choose k_2 elements $= \binom{n-k_1}{k_2}$. Continuing this reasoning, number of ways to split the set is:

$$\binom{n}{k_1}\binom{n-k_1}{k_2}\ldots\binom{n-\sum_{i=1}^{m-1}k_i}{k_m} = \frac{n!}{k_1!\, k_2!\, \ldots k_m!}$$

Generalization to Multinomials - Example

Q: Consider strings of size 5 to be formed from the letters $\{a, b, c\}$. Valid strings include: *aaaaa*, *abcba*, *bacba*, *cbcbc*. Calculate:

- Total number of such strings? Ans: $3^5 = 243$
- Number of strings that contain exactly 1 a, 1 b and 3 c? Ans: $\frac{5!}{1!1!3!}$
- Number of strings that contain exactly 3 a, 2 b and 0 c? Ans: $\frac{5!}{3!2!0!}$
- Number of strings that contain exactly 2 a, 1 b and 0 c? Ans: 0

Generalization to Multinomials - Example

Q: Count the number of permutations of the letters in the word BOOKKEEPER.

We want to count sequences of the form $(1E, 1P, 2E, 1B, 1K, 1R, 2O, 1K) = EPEEBKROOK$. There is a bijection between such sequences and $(1, 2, 2, 3, 1, 1)$ split of $A = \{1, 2, \ldots, 10\}$ where A_1 is the set of positions of B's, A_2 is the set of positions of O's, A_3 is set of positions of K and so on.

For example, the above sequence maps to the following split:
({5}, {8,9}, { 6, 10}, { 1,3,4 }, { 2 }, { 7 })

Hence, the total number of sequences that can be formed from the letters in BOOKKEEPER = number of $(1, 2, 2, 3, 1, 1)$ splits of $A = [10] = \{1, 2, \ldots, 10\} = \frac{10!}{1!\,2!\,2!\,3!\,1!\,1!}$.

Q: Count the number of permutations of the letters in the word (i) ABBA (ii) A_1BBA_2 and (iii) $A_1B_1B_2A_2$? Ans: 6, 12, 24

Generalization to Multinomials - Example

Q: Suppose we are planning a 20 km walk, which should include 5 northward km, 5 eastward km, 5 southward km, and 5 westward km. We can move in steps of 1 km in any direction. For example, a valid walk is $(NENWSNSSENSWWESWEENW)$ that corresponds to 1 km north followed by 1 km east and so on. How many different walks are possible?

Ans: The set $A = \{1, 2, \ldots, 20\}$ needs to be split into 4 subsets N, S, E, W s.t. $|N| = |S| = |E| = |W| = 5$. Counting the number of walks = counting the number of sequences of the form $(3N, 5W, 4S, 4E, 2N, 1E, 1S)$ = number of ways to obtain an $(5, 5, 5, 5)$-split of set $\{1, 2, 3, \ldots 20\}$. The total number of walks = $\frac{20!}{(5!)^4}$.

Multinomial Theorem

For all $m, n \in \mathbb{N}$ and $z_1, z_2, \ldots z_m \in \mathbb{R}$,

$$(z_1 + z_2 + \ldots + z_m)^n = \sum_{\substack{(k_1, k_2, \ldots, k_m) \\ k_1 + k_2 + \ldots k_m = n}} \binom{n}{k_1, k_2, \ldots, k_m} z_1^{k_1} z_2^{k_2} \ldots z_m^{k_m}$$

where $\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! k_2! \ldots k_m!}$.

Example 1: If $m = 2$, $k_1 = k$, $k_2 = n - k$ and $z_1 = a$, $z_2 = b$, recover the Binomial theorem.

Example 2: If $n = 4$, $m = 3$, then the coefficient of abc^2 in $(a + b + c)^4$ is $\binom{4}{1,1,2} = \frac{4!}{1!1!2!}$.

Questions?

Inclusion-Exclusion Principle

Recall that if A, B, C are disjoint subsets, then, $|A \cup B \cup C| = |A| + |B| + |C|$ (this is the Sum rule from Lecture 2).

- For two general (not necessarily disjoint) sets A, B, $|A \cup B| = |A| + |B| - |A \cap B|$.

The last term fixes the "double counting".

- Similarly, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$.
- In general,

$$|\cup_{i=1,2,\ldots n} A_i| = \sum_i |A_i| - \sum_{i,j \text{ s.t. } 1 \le i < j \le n} |A_i \cap A_j| + \sum_{i,j,k \text{ s.t. } 1 \le i < j < k \le n} |A_i \cap A_j \cap A_k|$$

$$+ \ldots + (-1)^{n-1} | \cap_{i=1,2,\ldots n} A_i|$$

Inclusion-Exclusion Principle - Example

Q: Suppose there are 60 math majors, 200 EECS majors, and 40 physics majors. A student is allowed to double or even triple major. There are 4 math-EECS double majors, 3 math-physics double majors, 11 EECS-physics double majors and 2-triple majors. Note that (i) a student double-majoring in Math and EECS counts as both a Math and an EECS major and by definition, they are not majoring in Physics (ii) student triple majoring in Math, EECS and Physics is counted as an EECS, Math and Phyiscs major, but does not count as a double-major. What is the total number of students across these three departments?

If M, E, P are the sets of students majoring in math, EECS and physics respectively, then we wish to compute $|M \cup E \cup P| = |M| + |E| + |P| - |M \cap E| - |M \cap P| - |E \cap P| + |M \cap E \cap P|$ $=$ 300 - $|M \cap E| - |M \cap P| - |E \cap P| + |M \cap E \cap P|$.

$|M \cap E| = 4 + 2 = 6$, $|M \cap P| = 3 + 2 = 5$, $|P \cap E| = 11 + 2 = 13$. $|M \cap E \cap P| = 2$

$|M \cup E \cup P| = 300 - 6 - 5 - 13 + 2 = 278$.

Inclusion-Exclusion Principle - Example

Q: In how many permutations of the set $\{0, 1, 2, \ldots, 9\}$ do either 4 and 2, 0 and 4, or 6 and 0 appear consecutively? For example, in the following permutation $\underline{42}06789135$, 4 and 2 appear consecutively, but 6 and 0 do not (the order matters).

Let P_{42} be the set of sequences such that 4 and 2 appear consecutively. Similarly, we define P_{60} and P_{04}. So we want to compute
$|P_{42} \cup P_{60} \cup P_{04}| = |P_{42}| + |P_{60}| + |P_{04}| - |P_{42} \cap P_{60}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{04}| + |P_{42} \cap P_{60} \cap P_{04}|$.

Let us first compute $|P_{42}| = 9!$. Similarly, $|P_{60}| = |P_{04}| = 9!$.

What about intersections? $|P_{42} \cap P_{60}| =$ Number of sequences of the form $(42, 60, 1, 3, 5, 7, 8, 9) = 8!$. Similarly, $|P_{60} \cap P_{04}| = |P_{42} \cap P_{04}| = 8!$.

$|P_{42} \cap P_{60} \cap P_{04}| =$ Number of sequences of the form $(6042, 1, 3, 5, 7, 8, 9) = 7!$.

By the inclusion-exclusion principle, $|P_{42} \cup P_{60} \cup P_{04}| = 3 \times 9! - 3 \times 8! + 7!$.

Questions?