CMPT 210: Probability and Computing

Lecture 3

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Recap - Counting

Product Rule: For sets $A_1, A_2 \ldots, A_m$, $|A_1 \times A_2 \times \ldots \times A_m| = \prod_{i=1}^{m} |A_i|$ (E.g: Selecting one course each from every subject.)

Sum rule: If $A_1, A_2 \ldots A_m$ are disjoint sets, then, $|A_1 \cup A_2 \cup \ldots \cup A_m| = \sum_{i=1}^{m} |A_i|$ (E.g Number of rainy, snowy or hot days in the year).

Generalized product rule: If S is the set of length k sequences such that the first entry can be selected in n_1 ways, after the first entry is chosen, the second one can be chosen in n_2 ways, and so on, then $|S| = n_1 \times n_2 \times \ldots n_k$. (E.g Number of ways n people can be arranged in a line $= n!$)

Division rule: $f : A \to B$ is a k-to-1 function, then, $|A| = k|B|$. (E.g. For arranging people around a round table, f : seatings $\to$ arrangements is an n-to-1 function).

Number of ways to select size-k subsets from a size-n set $= n$ choose $k = \binom{n}{k} := \frac{n!}{k! \times (n-k)!}$.

Counting subsets (Combinations)

Q: Prove that $\binom{n}{k} = \binom{n}{n-k}$ - both algebraically (using the formula for $\binom{n}{k}$) and combinatorially (without using the formula)

Ans: Algebraically, $\binom{n}{k}$ is symmetric with respect to k and $n-k$. Combinatorially, number of ways of choosing elements to form a set of size k = number of ways of choosing $n-k$ elements to discard.

Q: Which is bigger? $\binom{8}{4}$ vs $\binom{8}{5}$? Ans: $\binom{8}{4} = 70$. $\binom{8}{5} = 56$

Counting subsets – Example

Q: How many m-bit binary sequences contain exactly k ones?

Consider set $A = \{1, \ldots, m\}$ and selecting S, a subset of size k. For example, say $m = 10, k = 3$ and $S = \{3, 7, 10\}$. S records the positions of the 1's, and can mapped to the sequence 001 0001 001. Similarly, every m-bit sequence with exactly k ones can be mapped to a subset S of size k. Hence, there is a bijection:
f : m-bit sequence with exactly k ones $\rightarrow$ subsets of size k from size m-set, and
$|m\text{-bit sequence with exactly } k \text{ ones}| = |\text{subsets of size } k| = \binom{m}{k}$.

Counting subsets – Example

Q: What is the number of n-bit binary sequences with at least k ones?

Ans: Set of n-bit binary sequences with at least k ones $=$ n-bit binary sequences with exactly k ones $\cup$ n-bit binary sequences with exactly $k+1$ ones $\cup \ldots \cup$ n-bit binary sequences with exactly n ones. By the sum rule for disjoint sets, number of n-bit binary sequences with at least k ones $= \sum_{i=k}^{n} \binom{n}{i}$.

Q: What is the number of n-bit binary sequences with less than k ones?

Ans: $\sum_{i=0}^{k-1} \binom{n}{i}$

Q: What is the total number of n-bit binary sequences?

Ans: 2^n

Total number of n-bit binary sequences $=$ number of n-bit binary sequences with at least k ones $+$ number of n-bit binary sequences with less than k ones.
Combining the above answers, we can conclude that, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Have recovered a special case of the binomial theorem!

Binomial Theorem

For all $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Example: If $a = b = 1$, then $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ (result from previous slide).

If $n = 2$, then $(a+b)^2 = \binom{2}{0} a^2 + \binom{2}{1} ab + \binom{2}{2} b^2 = a^2 + 2ab + b^2$.

Q: What is the coefficient of the terms with ab^3 and $a^2 b^3$ in $(a+b)^4$? Ans: $\binom{4}{1} = \binom{4}{3}$, 0.

Q: For $a, b > 0$, what is the coefficient of $a^{2n-7} b^7$ and $a^{2n-8} b^8$ in $(a+b)^{2n} + (a-b)^{2n}$?

Ans: $(a+b)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} a^{2n-k} b^k$,
$(a-b)^{2n} = -\sum_{k=0}^{2n} \binom{2n}{k} a^{2n-k} b^k \mathcal{I}\{\text{k is odd}\} + \sum_{k=0}^{2n} \binom{2n}{k} a^{2n-k} b^k \mathcal{I}\{\text{k is even}\}$.
$(a+b)^{2n} + (a-b)^{2n} = 2\sum_{k=0}^{2n} \binom{2n}{k} a^{2n-k} b^k \mathcal{I}\{\text{k is even}\}$. Hence, coefficient of $a^{2n-7} b^7 = 0$, coefficient of $a^{2n-8} b^8 = 2\binom{2n}{8}$.

Counting Sets – using a bijection

Q: Suppose we want to buy 10 donuts. There are 5 donut varieties – chocolate, lemon-filled, sugar, glazed, plain. What is the number of ways to select the 10 donuts?

Let A be the set of ways to select the 10 donuts. Each element of A is a potential selection. For example, 4 chocolate, 3 lemon, 0 sugar, 2 glazed and 1 plain.

Let's map each way to a string as follows: $\underbrace{0000}_{\text{chocolate}}\ \underbrace{000}_{\text{lemon}}\ \underbrace{}_{\text{sugar}}\ \underbrace{00}_{\text{glazed}}\ \underbrace{0}_{\text{plain}}$.

Lets fix the ordering – chocolate, lemon, sugar, glazed and plain, and abstract this out further to get the sequence: 0000 1 000 1 1 00 1 0. Hence, each way of choosing donuts is mapped to a binary sequence of length 14 with exactly 4 ones. Now, let B be all 14-bit sequences with exactly 4 ones. An element of B is 11110000000000.

Q: The above sequence corresponds to what donut order? Ans: All plain donuts.

For every way to select donuts, we have an equivalent sequence in B. And every sequence in B implies a unique way to select donuts. Hence, the mapping from $A \to B$ is a bijective function.

Counting Sets – using a bijection

Hence, $|A| = |B|$, meaning that we have reduced the problem of counting the number of ways to select donuts to counting the number of 14-bit sequences with exactly 4 ones. This is equal to counting the number of subsets $= \binom{14}{4} = 1001$.

General result: The number of ways to choose n elements with k available varieties is equal to the number of $n+k-1$-bit binary sequences with exactly $k-1$ ones. This is equal to $\binom{n+k-1}{k-1}$.

Q: There are 2 donut varieties – chocolate and lemon-filled. Suppose we want to buy only 2 donuts. Use the above result to count the number of ways in which we can select the donuts? What are these ways?

Ans: Since $n = 2$, $k = 2$, we want to count the sequences with exactly 1 one in 3-bit sequences. $\{(0,0,1),(1,0,0),(0,1,0)\}$.

Q: In the above example, I want at least one chocolate donut. What are the types of acceptable 3-bit sequences with this criterion? How many ways can we do this?

Ans: We want to count the number of 3-bit sequences that start with zero and have exactly 1 one in them. So $\{(0,1,0),(0,0,1)\}$.

Questions?