CMPT 210: Probability and Computing

Lecture 17

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Recap

- **Expectation**/mean of a random variable R is denoted by $\mathbb{E}[R]$ and "summarizes" its distribution. Formally, $\mathbb{E}[R] := \sum_{\omega \in \mathcal{S}} \Pr[\omega]\, R[\omega]$
- **Alternate definition of expectation**: $\mathbb{E}[R] = \sum_{x \in \text{Range}(R)} x \Pr[R = x]$.
- **Linearity of Expectation**: For n random variables $R_1, R_2, \ldots, R_n$ and constants $a_1, a_2, \ldots, a_n$, $\mathbb{E}\left[\sum_{i=1}^{n} a_i R_i\right] = \sum_{i=1}^{n} a_i\, \mathbb{E}[R_i]$.
- **Expectation for Common Distributions**:
  - If $R \sim \text{Bernoulli}(p)$, $\mathbb{E}[R] = p$. *Example*: When tossing a coin, if R is the random variable equal to 1 if we get a heads.
  - If $R \sim \text{Uniform}(\{v_1, \ldots, v_n\})$, $\mathbb{E}[R] = \frac{v_1 + v_2 + \ldots + v_n}{n}$. *Example*: When throwing an n-sided dice with numbers $v_1, \ldots v_n$, if R is the random variable equal to the number.
  - If $R \sim \text{Bin}(n, p)$, $\mathbb{E}[R] = np$. *Example*: When tossing n independent coins, if R is the random variable equal to the number of heads.
  - If $R \sim \text{Geo}(p)$, $\mathbb{E}[R] = \frac{1}{p}$. *Example*: When tossing a coin repeatedly, if R is the random variable equal to the number of tosses required to get the first heads.

Conditional Expectation

- Similar to probabilities, expectations can be conditioned on some event.
- **Definition**: For random variable R, the expected value of R conditioned on an event A is:

$$\mathbb{E}[R|A] := \sum_{x \in \text{Range}(R)} x \; \Pr[R = x|A]$$

Q: If we throw a standard dice and define R to be the random variable equal to the number that comes up, what is the expected value of R given that the number is at most 4?

Let A be the event that the number is at most 4, i.e. $A = \{1, 2, 3, 4\}$.
$\Pr[R = 1|A] = \frac{\Pr[(R=1)\cap A]}{\Pr[A]} = \frac{\Pr[R=1]}{\Pr[A]} = \frac{1/6}{4/6} = 1/4$.
Similarly, $\Pr[R = 2|A] = \Pr[R = 3|A] = \Pr[R = 4|A] = \frac{1}{4}$ and $\Pr[R = 5|A] = \Pr[R = 6|A] = 0$.

$$\mathbb{E}[R|A] = \sum_{x \in \{1,2,3,4\}} x \; \Pr[R = x|A] = \frac{1}{4}[1 + 2 + 3 + 4] = \frac{5}{2}.$$

Q: What is the expected value of R given that the number is at least 4? Ans:
$\mathbb{E}[R|A] = \sum_{x \in \{4,5,6\}} x \; \Pr[R = x|A] = \frac{1}{3}[4 + 5 + 6] = 5$.

Law of Total Expectation

Law of Total Expectation: If R is a random variable $\mathcal{S} \to V$ and events $A_1, A_2, \ldots A_n$ form a partition of the sample space i.e. for all i, j, $A_i \cap A_j = \emptyset$ and $A_1 \cup A_2 \cup \ldots \cup A_n = \mathcal{S}$, then,

$$\mathbb{E}[R] = \sum_i \mathbb{E}[R|A_i] \Pr[A_i] \,.$$

Proof:

$$\mathbb{E}[R] = \sum_{x \in \text{Range}(R)} x \Pr[R = x] = \sum_{x \in \text{Range}(R)} x \sum_i \Pr[R = x|A_i] \Pr[A_i]$$

(Law of total probability)

$$= \sum_i \Pr[A_i] \sum_{x \in \text{Range}(R)} x \Pr[R = x|A_i] \qquad \text{(Rearranging the summations)}$$

$$\implies \mathbb{E}[R] = \sum_i \Pr[A_i] \, \mathbb{E}[R|A_i].$$

Conditional Expectation - Examples

Q: Suppose that 49.6% of the people in the world are male and the rest female. If the expected height of a randomly chosen male is 5 feet 11 inches, while the expected height of a randomly chosen female is 5 feet 5 inches, what is the expected height of a randomly chosen person?

Define H to be the random variable equal to the height (in feet) of a randomly chosen person.
Define M to be the event that the person is male and F the event that the person is female.
We wish to compute $\mathbb{E}[H]$ and we know that $\mathbb{E}[H|M] = 5 + \frac{11}{12}$ and $\mathbb{E}[H|F] = 5 + \frac{5}{12}$.
$\Pr[M] = 0.496$ and $\Pr[F] = 1 - 0.496 = 0.504$.
Hence, $\mathbb{E}[H] = \mathbb{E}[H|M] \Pr[M] + \mathbb{E}[H|F] \Pr[F] = \frac{71}{12}(0.496) + \frac{65}{12}(0.504)$.

Questions?

Independence of random variables

Definition: Random variables R_1 and R_2 are independent iff for *all* $x_1 \in \text{Range}(R_1)$ and $x_2 \in \text{Range}(R_2)$, events $[R_1 = x_1]$ and $[R_2 = x_2]$ are independent. Formally, we require,

$$\Pr[(R_1 = x_1) \cap (R_2 = x_2)] = \Pr[(R_1 = x_1)] \Pr[(R_2 = x_2)]$$

Q: Suppose we toss three independent, unbiased coins. Let C be r.v. equal to the number of heads that appear and M be the r.v. that is equal to 1 if all the coins match (else it is 0). Are random variables C and M independent?

$\mathcal{S} = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ Range$(C) = \{0, 1, 2, 3\}$ and Range$(M) = \{0, 1\}$. $\Pr[C = 3] = \frac{1}{8}$ and $\Pr[M = 1] = \frac{1}{4}$.
$\Pr[(C = 3) \cap (M = 1)] = \frac{1}{8} \neq \frac{1}{32} = \Pr[C = 3] \Pr[M = 1]$. Hence, C and M are not independent.

Independence - Examples

Q: Suppose we toss three independent, unbiased coins. If H_1 is the indicator r.v. equal to one if the first toss is a heads (else it is 0) and M be the r.v. that is equal to 1 if all the coins match (else it is 0), are H_1 and M independent?

The sample space is: $\mathcal{S} = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
$\Pr[H_1 = 1] = \Pr[H_1 = 0] = \frac{1}{2}$, $\Pr[M = 1] = \frac{1}{4}$, $\Pr[M = 0] = \frac{3}{4}$.
$\Pr[H_1 = 0 \cap M = 1] = \Pr[\{TTT\}] = \frac{1}{8} = \Pr[H_1 = 0] \Pr[M = 1]$.
$\Pr[H_1 = 1 \cap M = 1] = \Pr[\{HHH\}] = \frac{1}{8} = \Pr[H_1 = 1] \Pr[M = 1]$.
$\Pr[H_1 = 0 \cap M = 0] = \Pr[\{THH, THT, TTH\}] = \frac{3}{8} = \Pr[H_1 = 0] \Pr[M = 0]$.
$\Pr[H_1 = 1 \cap M = 0] = \Pr[\{HHT, HTH, HTT\}] = \frac{3}{8} = \Pr[H_1 = 1] \Pr[M = 0]$.
Hence, H_1 and M are independent.

Independence of random variables

Alternate definition of independence: Random variables R_1 and R_2 are independent iff for *all* $x_1 \in \text{Range}(R_1)$ and $x_2 \in \text{Range}(R_2)$,

$$\Pr[(R_1 = x_1)|(R_2 = x_2)] = \Pr[(R_1 = x_1)]$$
$$\Pr[(R_2 = x_2)|(R_1 = x_1)] = \Pr[(R_2 = x_2)]$$

Pairwise Independence: Similar to events, r.v's $R_1, R_2, R_3, \ldots R_n$ are *pairwise* independent iff for *any* pair R_i and R_j, for *all* $x \in \text{Range}(R_i)$ and $y \in \text{Range}(R_j)$, events $[R_i = x]$ and $[R_j = y]$ are pairwise independent implying that

$$\Pr[(R_i = x) \cap (R_j = y)] = \Pr[R_i = x] \Pr[R_j = y]$$

Alternatively, R_i and R_j are pairwise independent iff for *all* $x \in \text{Range}(R_i)$ and $y \in \text{Range}(R_j)$,

$$\Pr[(R_i = x)|(R_j = y)] = \Pr[(R_i = x)]$$
$$\Pr[(R_j = y)|(R_i = x)] = \Pr[(R_j = y)]$$

Independence of random variables

- Similar to events, random variables $R_1, R_2, \ldots, R_n$ are mutually independent if for *all* $x_1 \in \text{Range}(R_1), x_2 \in \text{Range}(R_2), \ldots, x_n \in \text{Range}(R_n)$, events $[R_1 = x_1], [R_2 = x_2], \ldots [R_n = x_n]$ are mutually independent.

Mutual Independence of events: A set of events is said to be mutually independent if the probability of each event in the set is the same no matter which subset of events has occurred. For events E_1, E_2 and E_3 to be mutually independent, all the following equalities should hold:

$$\Pr[E_1 \cap E_2] = \Pr[E_1]\Pr[E_2] \quad \Pr[E_1 \cap E_3] = \Pr[E_1]\Pr[E_3]$$
$$\Pr[E_2 \cap E_3] = \Pr[E_2]\Pr[E_3] \quad \Pr[E_1 \cap E_2 \cap E_3] = \Pr[E_1]\Pr[E_2]\Pr[E_3].$$

Alternatively, (i) $\forall i$ and $j \neq i$, $\Pr[E_i | E_j] = \Pr[E_i]$ and (ii) $\forall i$ and $j, k \neq i$, $\Pr[E_i | E_j \cap E_k] = \Pr[E_i]$.

- For 2 r.v's R_1 and R_2, mutual independence and pairwise independence are equivalent.
- For more than 2 r.v's $R_1, R_2, \ldots, R_n$, mutual independence implies pairwise independence.