CMPT 210: Probability and Computing

Lecture 15

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Recap

Bernoulli Distribution: $f_p(0) = 1 - p$, $f_p(1) = p$. *Example*: When tossing a coin such that $\Pr[\text{heads}] = p$, random variable R is equal to 1 if we get a heads (and equal to 0 otherwise). In this case, $R \sim \text{Ber}(p)$.

Uniform Distribution: If $R : \mathcal{S} \to V$, then for all $v \in V$, $f(v) = 1/|V|$. *Example*: When throwing an n-sided die, random variable R is the number that comes up on the die. $V = \{1, 2, \ldots, n\}$. In this case, $R \sim \text{Uniform}(\{1, 2, \ldots, n\})$.

Binomial Distribution: $f_{n,p}(k) = \binom{n}{k} p^k (1-p)^{n-k}$. *Example*: When tossing n independent coins such that $\Pr[\text{heads}] = p$, random variable R is the number of heads in n coin tosses. In this case, $R \sim \text{Bin}(n, p)$.

Geometric Distribution: $f_p(k) = (1-p)^{k-1} p$. *Example*: When repeatedly tossing a coin such that $\Pr[\text{heads}] = p$, random variable R is the number of tosses needed to get the first heads. In this case, $R \sim \text{Geo}(p)$.

Distributions - Examples

Q: It is known that disks produced by a certain company will be defective with probability 0.01 independently of each other. The company sells the disks in packages of 10 and offers a money-back guarantee that at most 1 of the 10 disks is defective (the package can be returned if there is more than 1 defective disk). If someone buys three packages, what is the probability that exactly one of them will be returned?

Let F be the event that someone bought 3 packages and exactly one of them is returned.

Answer 1: Let E_i be the event that package i is returned. From the previous question, we know that $\Pr[E_i] = \Pr[\text{Package } i \text{ has more than 1 defective disk}] \approx 0.005$.

$$F = (E_1 \cap E_2^c \cap E_3^c) \cup (E_1^c \cap E_2^c \cap E_3) \cup (E_1^c \cap E_2 \cap E_3^c)$$

$$\Pr[F] = \Pr[E_1](1 - \Pr[E_2])(1 - \Pr[E_3]) + (1 - \Pr[E_1])(1 - \Pr[E_2]) \Pr[E_3] + \dots$$

$$\Pr[F] \approx 3 \times (0.005)(0.995)(0.995) \approx 0.015.$$

Answer 2: Let Y be the random variable corresponding to the number of packages returned. Y follows the Binomial distribution $\text{Bin}(3, 0.05)$ and we wish to compute $\Pr[F] = \Pr[Y = 1] \approx \binom{3}{1}(0.005)^1(0.995)^2 \approx 0.015.$

Q: You are randomly and independently throwing darts. The probability that you hit the bullseye in throw i is p. Once you hit the bullseye you win and can go collect your reward. (a) What is the probability that you win in exactly k throws? (b) What is the probability you win in less than k throws?

(a) The number of throws (T) to hit the bullseye and win follows a geometric distribution $\text{Geo}(p)$ and we wish to compute $\Pr[T = k]$. Using the PDF for the Geometric distribution, this is equal to $(1-p)^{k-1}\, p$.

(b) **Answer 1**: If E is the event that we win in less than k throws,
$\Pr[E] = \Pr[T < k] = \sum_{i=1}^{k-1} \Pr[T = i] = p \sum_{i=1}^{k-1} (1-p)^{i-1} = 1 - (1-p)^{k-1}$.

Answer 2:
$\Pr[E] = 1 - \Pr[E^c] = 1 - \Pr[\text{do not hit the bullseye in } k-1 \text{ throws}] = 1 - (1-p)^{k-1}$.

Expectation of Random Variables

Recall that a random variable R is a total function from $\mathcal{S} \to V$.

Definition: Expectation of R is denoted by $\mathbb{E}[R]$ and "summarizes" its distribution. Formally,

$$\mathbb{E}[R] := \sum_{\omega \in \mathcal{S}} \Pr[\omega]\, R[\omega]$$

$\mathbb{E}[R]$ is also known as the "expected value" or the "mean" of the random variable R.

Q: We throw a standard dice, and define R to be the r.v. equal to the number that comes up. Calculate $\mathbb{E}[R]$.

$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ and for $\omega \in \mathcal{S}$, $R[\omega] = \omega$. Since this is a uniform probability space, $\Pr[\{1\}] = \Pr[\{2\}] = \ldots = \Pr[\{6\}] = \frac{1}{6}$.
$\mathbb{E}[R] = \sum_{\omega \in \mathcal{S}} \Pr[\omega]\, R[\omega] = \sum_{\omega \in \{1,2,\ldots,6\}} \Pr[\omega]\, \omega = \frac{1}{6}[1 + 2 + 3 + 4 + 5 + 6] = \frac{7}{2}$.

• A r.v. does not necessarily achieve its expected value. Intuitively, consider doing the "experiment" (throw a dice and record the number) multiple times This average of the numbers we record will tend to $\mathbb{E}[R]$ as the number of experiments becomes large.

Q: Let $T := {}^{1}/_{R}$. Is $\mathbb{E}[T] = {}^{1}/_{\mathbb{E}[R]}$? Ans: No. ${}^{1}/_{\mathbb{E}[R]} = 2/7$, $\mathbb{E}[T] = \frac{49}{120} \neq {}^{1}/_{\mathbb{E}[R]}$

Expectation of Random Variables

Alternate definition: $\mathbb{E}[R] = \sum_{x \in \text{Range}(R)} x \Pr[R = x]$.

Proof:

$$\mathbb{E}[R] = \sum_{\omega \in \mathcal{S}} \Pr[\omega]\, R[\omega] = \sum_{x \in \text{Range}(R)} \sum_{\omega | R(\omega) = x} \Pr[\omega]\, R[\omega] = \sum_{x \in \text{Range}(R)} \sum_{\omega | R(\omega) = x} \Pr[\omega]\, x$$

$$= \sum_{x \in \text{Range}(R)} x \left[\sum_{\omega | R(\omega) = x} \Pr[\omega] \right] = \sum_{x \in \text{Range}(R)} x \Pr[R = x]$$

- This definition does not depend on the sample space.

Q: We throw a standard dice, and define R to be the random variable equal to the number that comes up. Calculate $\mathbb{E}[R]$.

Range(R) $= \{1,2,3,4,5,6\}$. R has a uniform distribution i.e. $\Pr[R = 1] = \ldots = \Pr[R = 6] = \frac{1}{6}$.
Hence, $\mathbb{E}[R] = \frac{1}{6}[1 + \ldots + 6] = \frac{7}{2}$.