CMPT 210: Probability and Computing

Lecture 14

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Recap

- **Random variable**: A random "variable" R on a probability space is a total function whose domain is the sample space $\mathcal{S}$. The codomain is denoted by V (usually a subset of the real numbers), meaning that $R : \mathcal{S} \to V$. A r.v partitions the sample space into several blocks.

Example: Suppose we toss three independent, unbiased coins. In this case, $\mathcal{S} = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. C is a random variable equal to the number of heads that appear such that $C : \mathcal{S} \to \{0, 1, 2, 3\}$. $C(HHT) = 2$.

- For r.v. R, for all $i \in \text{Range}(R)$, the event $[R = i] = \{\omega \in \mathcal{S} | R(\omega) = i\}$. For any r.v. R, $\sum_{i \in \text{Range(R)}} \Pr[R = i] = 1$.

Example: $[C = 2] = \{HHT, HTH, THH\}$ and $\Pr[C = 2] = \frac{3}{8}$.
$\sum_{i \in \text{Range(C)}} \Pr[C = i] = \Pr[C = 0] + \Pr[C = 1] + \Pr[C = 2] + \Pr[C = 3] = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1.$

Recap

- **Indicator Random Variable**: An indicator random variable corresponding to an event E is denoted as $\mathcal{I}_E$ and is defined such that for $\omega \in E$, $\mathcal{I}_E[\omega] = 1$ and for $\omega \notin E$, $\mathcal{I}_E[\omega] = 0$.

Example: When throwing two dice, if E is the event that both throws of the dice result in a prime number, then $\mathcal{I}_E((2,4)) = 0$ and $\mathcal{I}_E((2,3)) = 1$.

- **Probability density function (PDF)**: Let R be a r.v. with codomain V. The probability density function of R is the function $\mathrm{PDF}_R : V \to [0,1]$, such that $\mathrm{PDF}_R[x] = \Pr[R = x]$ if $x \in$ Range(R) and equal to zero if $x \notin$ Range(R).

- **Cumulative distribution function (CDF)**: The cumulative distribution function of R is the function $\mathrm{CDF}_R : \mathbb{R} \to [0,1]$, such that $\mathrm{CDF}_R[x] = \Pr[R \leq x]$.

Importantly, neither PDF_R nor CDF_R involves the sample space of an experiment.

Example: If we flip three coins, and C counts the number of heads, then
$\mathrm{PDF}_C[0] = \Pr[C = 0] = \frac{1}{8}$, and
$\mathrm{CDF}_C[2.3] = \Pr[C \leq 2.3] = \Pr[C = 0] + \Pr[C = 1] + \Pr[C = 2] = \frac{7}{8}$.

Distributions

Many random variables turn out to have the same PDF and CDF. In other words, even though R and T might be different random variables on different probability spaces, it is often the case that $\text{PDF}_R = \text{PDF}_T$. Hence, by studying the properties of such PDFs, we can study different random variables and experiments.

- **Distribution** over a random variable can be fully specified using the cumulative distribution function (CDF) (usually denoted by F). The corresponding probability density function (PDF) is denoted by f.
- **Common Discrete Distributions** in Computer Science:
  - Bernoulli Distribution
  - Uniform Distribution
  - Binomial Distribution
  - Geometric Distribution

Bernoulli Distribution

Canonical Example: We toss a biased coin such that the probability of getting a heads is p. Let R be the random variable such that $R = 1$ when the coin comes up heads and $R = 0$ if the coin comes up tails. R follows the Bernoulli distribution.

PDF$_R$ for Bernoulli distribution: $f\colon \{0,1\} \to [0,1]$ meaning that Bernoulli random variables take values in $\{0,1\}$. It can be fully specified by the "probability of success" (of an experiment) p (probability of getting a heads in the example). Formally, PDF$_R$ is given by:

$$f(1) = p \quad ; \quad f(0) = q := 1 - p.$$

In the example, $\Pr[R = 1] = f(1) = p = \Pr[\text{event that we get a heads}]$.

CDF$_R$ for Bernoulli distribution: $F : \mathbb{R} \to [0,1]$:

$$\begin{aligned} F(x) &= 0 && \text{(for } x < 0) \\ &= 1 - p && \text{(for } 0 \leq x < 1) \\ &= 1 && \text{(for } x \geq 1) \end{aligned}$$

Uniform Distribution

Canonical Example: We roll a standard die. Let R be the random variable equal to the number that shows up on the die. R follows the uniform distribution.

A random variable R that takes on each possible value in its codomain V with the same probability is said to be uniform.

PDF$_R$ for Uniform distribution: $f : V \to [0,1]$ such that for all $v \in V$, $f(v) = {}^{1}/_{|V|}$. In the example, $f(1) = f(2) = \ldots = f(6) = \frac{1}{6}$.

CDF$_R$ for Uniform distribution: For n elements in V arranged in increasing order – $(v_1, v_2, \ldots, v_n)$, the CDF is:

$$
\begin{aligned}
F(x) &= 0 && \text{(for } x < v_1\text{)} \\
&= {}^{k}/_{n} && \text{(for } v_k \le x < v_{k+1}\text{)} \\
&= 1 && \text{(for } x \ge v_n\text{)}
\end{aligned}
$$

Q: If X has a Bernoulli distribution, when is X also uniform? Ans: When $p = 1/2$

Binomial Distribution

Canonical Example: We toss n biased coins independently. The probability of getting a heads for each coin is p. Let R be the random variable equal to the number of heads in the n coin tosses. R follows the Binomial distribution.

PDF$_R$ for Binomial distribution: $f : \{0, 1, 2, \ldots, n\} \to [0, 1]$. For $k \in \{0, 1, \ldots, n\}$, $f(k) = \binom{n}{k} p^k (1-p)^{n-k}$.

Proof: Let E_k be the event we get k heads. Let A_i be the event we get a heads in toss i.

$$E_k = (A_1 \cap A_2 \ldots A_k \cap A_{k+1}^c \cap A_{k+2}^c \cap \ldots \cap A_n^c) \cup (A_1^c \cap A_2 \ldots A_k \cap A_{k+1} \cap A_{k+2}^c \cap \ldots \cap A_n^c) \cup \ldots$$

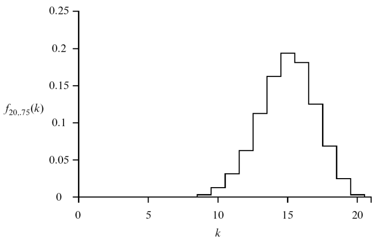
$$\begin{aligned}
\Pr[E_k] &= \Pr[(A_1 \cap A_2 \ldots A_k \cap A_{k+1}^c \cap A_{k+2}^c \cap \ldots \cap A_n^c)] + \Pr[A_1^c \cap A_2 \ldots A_k \cap A_{k+1} \cap \ldots \cap] + \ldots \\
&= \Pr[A_1] \Pr[A_2] \Pr[A_k] \Pr[A_{k+1}^c] \Pr[A_{k+2}^c] \ldots \Pr[A_n^c] + \ldots \qquad \text{(Independence of tosses)} \\
&= p^k (1-p)^{n-k} + p^k (1-p)^{n-k} + \ldots
\end{aligned}$$

$$\implies \Pr[E_k] = \binom{n}{k} p^k (1-p)^{n-k}$$

(Number of terms = number of ways to choose the k tosses that result in heads = $\binom{n}{k}$)

Binomial Distribution

For the Binomial distribution, $\mathrm{PDF}_R(k) = \binom{n}{k} p^k (1-p)^{n-k}$.



Q: Prove that $\sum_{k \in \mathrm{Range(R)}} \mathrm{PDF}_R[k] = 1$.
By the Binomial Theorem, $\sum_{k \in \mathrm{Range(R)}} \mathrm{PDF}_R[k] = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + 1 - p)^n = 1$.

CDF$_R$ for Binomial distribution: $F : \mathbb{R} \to [0, 1]$:

$$\begin{aligned} F(x) &= 0 && (\text{for } x < 0) \\ &= \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} && (\text{for } k \le x < k+1) \\ &= 1. && (\text{for } x \ge n) \end{aligned}$$

Geometric Distribution

Canonical Example: We toss a biased coin independently multiple times. The probability of getting a heads is p. Let R be the random variable equal to the number of tosses needed to get the first heads. R follows the geometric distribution.

PDF$_R$ for Geometric distribution: $f : \{1, 2, \ldots\} \to [0, 1]$. For $k \in \{1, 2, \ldots, \infty\}$, $f(k) = (1-p)^{k-1}\, p$.

Proof: Let E_k be the event that we need k tosses to get the first heads. Let A_i be the event that we get a heads in toss i.

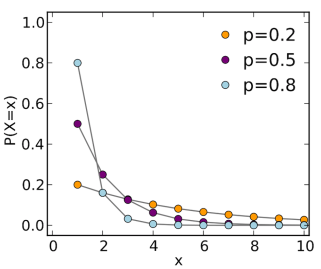
$$
\begin{aligned}
E_k &= A_1^c \cap A_2^c \cap \ldots \cap A_k \\
\Pr[E_k] &= \Pr[A_1^c \cap A_2^c \cap \ldots \cap A_k] = \Pr[A_1^c]\Pr[A_2^c]\ldots\Pr[A_k] \quad \text{(Independence of tosses)} \\
\implies \Pr[E_k] &= (1-p)^{k-1} p
\end{aligned}
$$

Q: Prove that $\sum_{k \in \text{Range(R)}} \text{PDF}_R[k] = 1$.

By the sum of geometric series, $\sum_{k \in \text{Range}(R)} \text{PDF}_R[k] = \sum_{k=1}^{\infty} (1-p)^{k-1} p = \frac{p}{1-(1-p)} = 1$.

Geometric Distribution

For the Geometric distribution, $\mathrm{PDF}_R(k) = (1-p)^{k-1}p$.



CDF$_R$ for Geometric distribution: $F : \mathbb{R} \to [0,1]$:

$$F(x) = 0 \qquad \text{(for } x < 1\text{)}$$

$$= \sum_{i=1}^{k} (1-p)^{i-1} p \qquad \text{(for } k \le x < k+1\text{)}$$

Questions?

Q: It is known that disks produced by a certain company will be defective with probability 0.01 independently of each other. The company sells the disks in packages of 10 and offers a money-back guarantee that at most 1 of the 10 disks is defective (the package can be returned if there is more than 1 defective disk). What proportion of packages is returned?

Let X be the random variable corresponding to the number of defective disks in a package. Let E be the event that the package is returned. We wish to compute $\Pr[E] = \Pr[X > 1]$. X follows the Binomial distribution $\mathrm{Bin}(10, 0.01)$. Hence,

$$\begin{aligned}\Pr[E] = \Pr[X > 1] = 1 - \Pr[X \leq 1] &= 1 - \Pr[X = 0] - \Pr[X = 1] \\ &= 1 - \binom{10}{0}(0.99)^{10} - \binom{10}{1}(0.99)^{9}(0.01)^{1} \approx 0.005\end{aligned}$$