CMPT 210: Probability and Computing

Lecture 13

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October 24, 2024

Recap

Random variable: A random "variable" R on a probability space is a total function whose domain is the sample space $\mathcal{S}$. The codomain is denoted by V (usually a subset of the real numbers), meaning that $R : \mathcal{S} \to V$.

Example: Suppose we toss three independent, unbiased coins. In this case, $\mathcal{S} = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. C is a random variable equal to the number of heads that appear such that $C : \mathcal{S} \to \{0, 1, 2, 3\}$. $C(HHT) = 2$.

Random Variables and Events

Indicator Random Variable: An indicator random variable maps every outcome to either 0 or 1.

Example: Suppose we throw two standard dice, and define M to be the random variable that is 1 iff both throws of the dice produce a prime number, else it is 0.

$M : \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\} \to \{0,1\}$. $M((2,3)) = 1$, $M((3,6)) = 0$.

• An indicator random variable partitions the sample space into those outcomes mapped to 1 and those outcomes mapped to 0.

Example: When throwing two dice, if E is the event that both throws of the dice result in a prime number, then random variable $M = 1$ iff event E happens, else $M = 0$.

• The indicator random variable corresponding to an event E is denoted as $\mathcal{I}_E$, meaning that for $\omega \in E$, $\mathcal{I}_E[\omega] = 1$ and for $\omega \notin E$, $\mathcal{I}_E[\omega] = 0$. In the above example, $M = \mathcal{I}_E$ and since $(2,4) \notin E$, $M((2,4)) = 0$ and since $(3,5) \in E$, $M((3,5)) = 1$.

Random Variables and Events

- In general, a random variable that takes on several values partitions $\mathcal{S}$ into several blocks.

Example: When we toss a coin three times, and define C to be the r.v. that counts the number of heads, C partitions $\mathcal{S}$ as follows: $\mathcal{S} = \{\underbrace{HHH}_{C=3}, \underbrace{HHT, HTH, THH}_{C=2}, \underbrace{HTT, THT, TTH}_{C=1}, \underbrace{TTT}_{C=0}\}$.

- Each block is a subset of the sample space and is therefore an event. For example, $[C = 2]$ is the event that the number of heads is two and consists of the outcomes $\{HHT, HTH, THH\}$.

- Since it is an event, we can compute its probability i.e.

$\Pr[C = 2] = \Pr[\{HHT, HTH, THH\}] = \Pr[\{HHT\}] + \Pr[\{HTH\}] + \Pr[\{THH\}]$. Since this is a uniform probability space, $\Pr[\omega] = \frac{1}{8}$ for $\omega \in \mathcal{S}$ and hence $\Pr[C = 2] = \frac{3}{8}$.

Q: What is $\Pr[C = 0]$, $\Pr[C = 1]$ and $\Pr[C = 3]$? Ans: $\frac{1}{8}, \frac{3}{8}, \frac{1}{8}$

Q: What is $\sum_{i=0}^{3} \Pr[C = i]$? Ans: 1

- Since a random variable R is a total function that maps every outcome in $\mathcal{S}$ to some value in the codomain, $\sum_{i \in \text{Range of R}} \Pr[R = i] = \sum_{i \in \text{Range of R}} \sum_{\omega \text{ s.t. } R(\omega)=i} \Pr[\omega] = \sum_{\omega \in \mathcal{S}} \Pr[\omega] = 1$.

Back to throwing dice

Q: Suppose we throw two standard dice one after the other. Let us define R to be the random variable equal to the sum of the dice. What are the outcomes in the event $[R = 2]$? Ans: $\{(1, 1)\}$

Q: What is $\Pr[R = 4]$, $\Pr[R = 9]$? Ans: $\frac{3}{36}$, $\frac{4}{36}$

Q: If M is the indicator random variable equal to 1 iff both throws of the dice produces a prime number, what is $\Pr[M = 1]$? Ans: $\frac{9}{36}$

Distribution Functions

Probability density function (PDF): Let R be a random variable with codomain V. The probability density function of R is the function $\text{PDF}_R : V \to [0,1]$, such that $\text{PDF}_R[x] = \Pr[R = x]$ if $x \in \text{Range(R)}$ and equal to zero if $x \notin \text{Range(R)}$.

$\sum_{x \in V} \text{PDF}_R[x] = \sum_{x \in \text{Range(R)}} \Pr[R = x] = 1$.

Cumulative distribution function (CDF): If the codomain is a subset of the real numbers, then the cumulative distribution function is the function $\text{CDF}_R : \mathbb{R} \to [0,1]$, such that $\text{CDF}_R[x] = \Pr[R \leq x]$.

- Importantly, neither PDF_R nor CDF_R involves the sample space of an experiment.

Example: If we flip three coins, and C counts the number of heads, then
$\text{PDF}_C[0] = \Pr[C = 0] = \frac{1}{8}$, and
$\text{CDF}_C[2.3] = \Pr[C \leq 2.3] = \Pr[C = 0] + \Pr[C = 1] + \Pr[C = 2] = \frac{7}{8}$.

Q: What is $\text{CDF}_C[5.8]$? Ans: 1.

- For a general random variable R, as $x \to \infty$, $\text{CDF}_R[x] \to 1$ and $x \to -\infty$, $\text{CDF}_R[x] \to 0$.

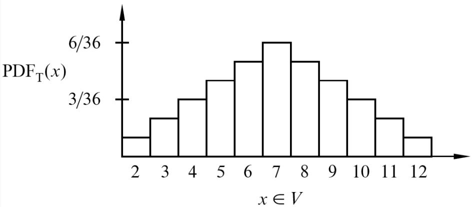
Back to throwing dice

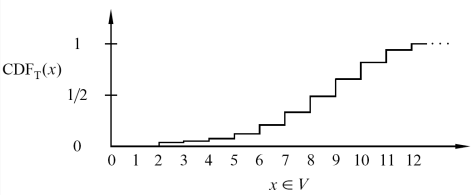
Q: Suppose we throw two standard dice one after the other. Let us define T to be the random variable equal to the sum of the dice. Plot PDF_T and CDF_T

Recall that $T : \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\} \to V$ where $V = \{2, 3, 4, \ldots 12\}$.

$\text{PDF}_T : V \to [0, 1]$ and $\text{CDF}_T : \mathbb{R} \to [0, 1]$.

For example, $\text{PDF}_T[4] = \Pr[T = 4] = \frac{3}{36}$ and $\text{PDF}_T[12] = \Pr[T = 12] = \frac{1}{36}$.





Questions?