CMPT 210: Probability and Computing

Lecture 9

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Recap

For events E and F, we wish to compute $\Pr[E|F]$, the probability of event E conditioned on F.

Approach 1: With conditioning, F can be interpreted as the *new sample space* such that for $\omega \notin F$, $\Pr[\omega|F] = 0$.

Approach 2: $\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$.

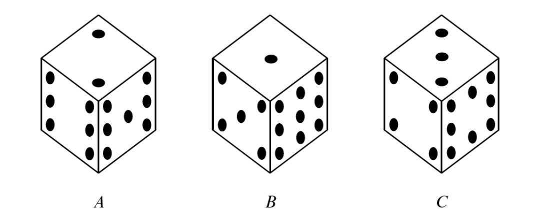
Multiplication Rule: For events $E_1, E_2, \ldots, E_n$,
$\Pr[E_1 \cap E_2 \ldots \cap E_n] = \Pr[E_1] \Pr[E_2|E_1] \Pr[E_3|E_1 \cap E_2] \ldots \Pr[E_n|E_1 \cap E_2 \cap \ldots E_{n-1}]$.

Tree Diagrams:

- Helpful in calculating probabilities in a sequential process (E.g. In the Monty Hall problem, the process is choose car location, choose door, reveal door).
- In a tree diagram, edge-weights correspond to conditional probabilities and leaf nodes correspond to outcomes.
- The probability of an outcome can be calculated by multiplying the relevant probabilities along a path.

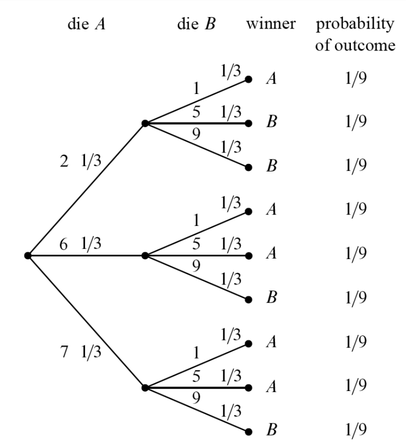
Conditional Probability - Examples

Let us play a game with three strange dice shown in the figure. Each player selects one die and rolls it once. The player with the lower value pays the other player $100. We can pick a die first, after which the other player can pick one of the other two.



Q: Suppose we choose die B because it has a 9, and the other player selects die A. What is the probability that we will win?

Conditional Probability - Examples



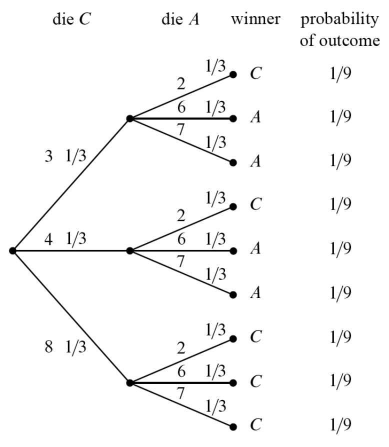
Identify Outcomes: Each leaf is an outcome and $\mathcal{S} = \{(2,1),(2,5),(2,9),(6,1),(6,5),(6,9),(7,1),(7,5),(7,9)\}$.

Identify Event: $E = \{(2,5),(2,9),(6,9),(7,9)\}$.

Compute probabilities: $\Pr[\text{Dice 1 is 6}] = \frac{1}{3}$.
$\Pr[(6,5)] = \Pr[\text{Dice 2 is 5} \ \cap \ \text{Dice 1 is 6}] = \Pr[\text{Dice 2 is 5} \mid \text{Dice 1 is 6}] \Pr[\text{Dice 1 is 6}] = \frac{1}{3}\frac{1}{3} = \frac{1}{9}$.

$\Pr[E] = \Pr[(2,5)] + \Pr[(2,9)] + \Pr[(6,9)] + \Pr[(7,9)] = \frac{4}{9}$.
Meaning that there is less than 50% chance of winning.

Q: We get another chance – this time we know that die A is good (since we lost to it previously), we choose die A and the other player chooses die C. What is our probability of winning?



Now, $E = \{(3,6),(3,7),(4,6),(4,7)\}$ and hence $\Pr[E] = \frac{4}{9}$. Meaning that there is less than 50% chance of winning.

Conditional Probability - Examples

We get yet another chance, and this time we choose die C, because we reason that die A is better than B, and C is better than A.

We can construct a similar tree diagram to show that the probability that we win is again $\frac{4}{9}$.

- A beats B with probability $\frac{5}{9}$ (first game).
- C beats A with probability $\frac{5}{9}$ (second game).
- B beats C with probability $\frac{5}{9}$ (third game).

Since A will beat B more often than not, and B will beat C more often than not, it seems like A ought to beat C more often than not, that is, the "beats more often" relation ought to be transitive. But this intuitive idea is false: whatever die we pick, the second player can pick one of the others and be likely to win. So picking first is actually a disadvantage!

This is the topic of some recent research and was covered in this article:
`https://www.quantamagazine.org/mathematicians-roll-dice-and-get-rock-paper-scissors-20230119/`

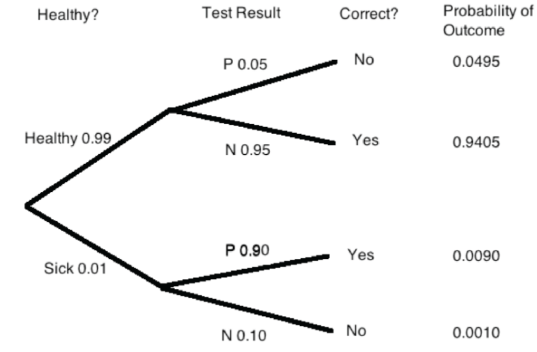
Conditional Probability - Examples

Q: A test for detecting cancer has the following accuracy – (i) If a person has cancer, there is a 10% chance that the test will say that the person does not have it. This is called a "false negative" and (ii) If a person does not have cancer, there is a 5% chance that the test will say that the person does have it. This is called a "false positive". For patients that have no family history of cancer, the incidence of cancer is 1%. Person X does not have any family history of cancer, but is detected to have cancer. What is the probability that the Person X does have cancer?

Conditional Probability - Examples

$\mathcal{S} = \{(Healthy, Positive), (Healthy, Negative), (Sick, Positive), (Sick, Negative)\}$.

A is the event that Person X has cancer. B is the event that the test is positive.



$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{\Pr[\{(S,P)\}]}{\Pr[\{(S,P),(H,P)\}]} = \frac{0.0090}{0.0090+0.0495} \approx 15.4\%.$$

Questions?

Conditional Probability

Conditional probability for complement events: For events E, F, $\Pr[E^c|F] = 1 - \Pr[E|F]$.

Proof: Since $E \cup E^c = \mathcal{S}$, for an event F such that $\Pr[F] \neq 0$,

$$
\begin{aligned}
(E \cup E^c) \cap F &= \mathcal{S} \cap F = F \\
(E \cup E^c) \cap F &= (E \cap F) \cup (E^c \cap F) && \text{(Distributive Law)} \\
\implies \Pr[(E \cap F) \cup (E^c \cap F)] &= \Pr[F]
\end{aligned}
$$

Since $E \cap F$ and $E^c \cap F$ are mutually exclusive events,

$$
\begin{aligned}
\Pr[E \cap F] + \Pr[E^c \cap F] = \Pr[F] &\implies \frac{\Pr[E^c \cap F]}{\Pr[F]} = 1 - \frac{\Pr[E \cap F]}{\Pr[F]} \\
&\implies \Pr[E^c|F] = 1 - \Pr[E|F] && \text{(By def. of conditional probability)}
\end{aligned}
$$

Bayes Rule

Bayes Rule: For events E and F if $\Pr[E] \neq 0$ and $\Pr[F] \neq 0$, then, $\Pr[F|E] = \frac{\Pr[E|F]\Pr[F]}{\Pr[E]}$.
Proof: Using the formula for conditional probability,

$$\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]} \quad ; \quad \Pr[F|E] = \frac{\Pr[F \cap E]}{\Pr[E]}$$

$$\implies \Pr[E \cap F] = \Pr[E|F]\Pr[F] \quad ; \quad \Pr[F \cap E] = \Pr[F|E]\Pr[E]$$

$$\implies \Pr[E|F]\Pr[F] = \Pr[F|E]\Pr[E]$$

$$\implies \Pr[F|E] = \frac{\Pr[E|F]\Pr[F]}{\Pr[E]}$$

Allows us to compute $\Pr[F|E]$ using $\Pr[E|F]$. Later in the course, we will see an application of the Bayes rule to machine learning.