CMPT 210: Probability and Computing

Lecture 7

Sharan Vaswani

January 30, 2024

Recap

Sample (outcome) space $\mathcal{S}$: Nonempty (countable) set of possible outcomes.

Outcome $\omega \in \mathcal{S}$: Possible "thing" that can happen.

Event E: Any subset of the sample space. An event E "happens" if the outcome ω (from some process) is in set E i.e. if $\omega \in E$.

Probability function on a sample space $\mathcal{S}$ is a total function $\Pr : \mathcal{S} \to [0, 1]$. For any $\omega \in \mathcal{S}$,

$$0 \leq Pr[\omega] \leq 1 \quad ; \quad \sum_{\omega \in \mathcal{S}} \Pr[\omega] = 1 \quad ; \quad \Pr[E] = \sum_{\omega \in E} \Pr[\omega]$$

Complement rule: $\Pr[E] = 1 - \Pr[E^c]$

Inclusion-Exclusion rule: For any two events E, F, $\Pr[E \cup F] = \Pr[E] + \Pr[F] - \Pr[E \cap F]$.

Uniform probability space: A probability space is said to be uniform if $\Pr[\omega]$ is the same for every outcome $\omega \in \mathcal{S}$. In this case, $\Pr[E] = \frac{|E|}{|\mathcal{S}|}$.

Birthday Principle

If there are n pigeons and d pigeonholes, then the probability that two pigeons occupy the same hole is $\geq 1 - \exp\left(-\frac{n(n-1))}{2d}\right)$

For $n = \lceil\sqrt{2d}\rceil$, probability that two pigeons occupy the same hole is about $1 - \frac{1}{e} \approx 0.632$.

Example: If we are randomly throwing $\lceil\sqrt{2d}\rceil$ balls into d bins, then the probability that two balls land in the same bin is around 0.632.

Later in the course, we will see applications of this principle to load balancing.

Conditional Probability

Conditioning is revising probabilities based on partial information (an event).

Q: Suppose we throw a "standard" dice, what is the probability of getting a 6 if we are told that the outcome is even?

We wish to compute Pr[we get a 6|the outcome is even] or Probability of getting a 6 *given* that the outcome is even or Probability of a 6 *conditioned on the event* that the outcome is even.

Sample space: $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$, Event: $E = \{6\}$. Additional information: Event $F = \{2, 4, 6\}$ has happened. With conditioning on F, *new sample space* $S' = F = \{2, 4, 6\}$. Since each outcome in $S' = \{2, 4, 6\}$ is equally likely, the new probability space is still a uniform probability space. Hence, conditioned on the event that the outcome is even, $\sum_{\omega \in S'} \Pr[\omega] = 1$ and $\Pr[\text{even number}] = \frac{1}{3}$ and $\Pr[\text{odd number}] = 0$. Hence, $\Pr[6] = \frac{1}{3}$.

Q: What is the probability of getting either a 3 or 6 if we are told that the outcome is even? $E = \{3, 6\}$, $F = \{2, 4, 6\}$. With conditioning, the new sample space $S' = \{2, 4, 6\}$. By the same reasoning as above, $\Pr[6] = \frac{1}{3}$ and $\Pr[3] = 0$. Hence, $\Pr[E] = \Pr[3] + \Pr[6] = \frac{1}{3}$.

Conditional Probability

Q: Suppose we throw two standard dice one after the other. What is the probability that the sum of the dice is 6?

Recall the sample space consists of tuples, i.e. $S = \{(1,1),(1,2),(1,3),\ldots,(6,6)\}$. The event E consists of outcomes such that the sum of the dice is 6. $E = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$. Since all outcomes are equally likely, this is a uniform probability space, and $\Pr[E] = \frac{|E|}{|S|} = \frac{5}{36}$.

Q: Suppose I tell you that the first dice came up 4. Given this information, what is the probability that the sum of the dice is 6?

Let F be the event that the first dice came up 4. We wish to compute $\Pr[E|F]$, the probability that the sum of the dice is 6 conditioned on the event that the first dice came up 4.

With conditioning, the new sample space $S' = F = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}$. For this new probability space, $E = \{(4,2)\}$. Since each outcome in S' is equally likely, $\Pr[E|F] = \frac{|E|}{|S'|} = \frac{1}{6}$.

Conditional Probability

Conditional Probability Rule: For two events E and F, $\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$, where $\Pr[F] \neq 0$.

Proof: By conditioning on F, the only outcomes we care about are in F i.e. for $\omega \notin F$, $\Pr[\omega|F] = 0$.

Since we want to compute the probability that event E happens, we care about the outcomes that are in E. Hence, the outcomes we care about lie in both E *and* F, meaning that $\omega \in E \cap F$.
$\implies \Pr[E|F] \propto \sum_{\omega \in (E \cap F)} \Pr[\omega]$. By definition of proportionality, for some constant $c > 0$, $\Pr[E|F] = c \sum_{\omega \in (E \cap F)} \Pr[\omega]$.

We know that $\Pr[F|F] = 1$ (probability of event F given that F has happened). Hence, $\Pr[F|F] = 1 = c \sum_{\omega \in F} \Pr[\omega] \implies c = \frac{1}{\sum_{\omega \in F} \Pr[\omega]}$.

Substituting the value of c,

$$\Pr[E|F] = \frac{\sum_{\omega \in (E \cap F)} \Pr[\omega]}{\sum_{\omega \in F} \Pr[\omega]} = \frac{\Pr[E \cap F]}{\Pr[F]}, \text{where } \Pr[F] \neq 0.$$

This formula gives an alternate way to compute conditional probabilities.

Back to throwing dice

Q: Suppose we throw a "standard" dice, what is the probability of getting a 6 if we are told that the outcome is even?

Sample space: $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ and $\Pr[1] = \Pr[2] = \ldots = \Pr[6] = \frac{1}{6}$.
Event: $E = \{6\}$. We are conditioning on $F = \{2, 4, 6\}$.
$\Pr[\text{we get a 6}|\text{the outcome is even}] = \Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$.
$E \cap F = \{6\}$. $\Pr[E \cap F] = \frac{1}{6}$. $\Pr[F] = \Pr[2] + \Pr[4] + \Pr[6] = \frac{1}{2}$.
Hence, $\frac{\Pr[E \cap F]}{\Pr[F]} = \frac{1/6}{1/2} = \frac{1}{3}$.

Q: What is the probability of getting either a 3 or 6 if we are told that the outcome is even? Ans: $E = \{3, 6\}$, $F = \{2, 4, 6\}$. $E \cap F = \{6\}$. $\Pr[E] = \Pr[6] = \frac{1}{6}$. $\Pr[F] = \frac{1}{2}$. Hence, $\Pr[E|F] = \frac{1}{3}$.

Q: Suppose we throw two standard dice one after the other. What is the probability that the sum of the dice is 6 given that the first dice came up 4? Ans: $S = \{(1, 1), (1, 2), (1, 3), \ldots, (6, 6)\}$,
$E = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$, $F = \{(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)\}$.
$\Pr[F] = \frac{6}{36}$, $E \cap F = \{(4, 2)\}$ $\Pr[E \cap F] = \frac{1}{36}$, $\Pr[E|F] = \frac{1/36}{6/36} = \frac{1}{6}$

Conditional Probability - Examples

Q: Suppose we select a card at random from a standard deck of 52 cards. What is the probability of getting:

- A spade conditioned on the event that I picked the red color Ans: 0
- A spade facecard conditioned on the event that I picked the black color Ans: $\frac{3}{26}$
- A black card conditioned on the event that I picked a spade facecard Ans: 1
- The queen of hearts given that I picked a queen Ans: $\frac{1}{4}$
- An ace given that I picked a spade Ans: $\frac{1}{13}$