CMPT 210: Probability and Computing

Lecture 3

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Recap - Counting

Product Rule: For sets $A_1, A_2 \ldots, A_m$, $|A_1 \times A_2 \times \ldots \times A_m| = \prod_{i=1}^{m} |A_i|$ (E.g: Selecting one course each from every subject.)

Sum rule: If $A_1, A_2 \ldots A_m$ are disjoint sets, then, $|A_1 \cup A_2 \cup \ldots \cup A_m| = \sum_{i=1}^{m} |A_i|$ (E.g Number of rainy, snowy or hot days in the year).

Generalized product rule: If S is the set of length k sequences such that the first entry can be selected in n_1 ways, after the first entry is chosen, the second one can be chosen in n_2 ways, and so on, then $|S| = n_1 \times n_2 \times \ldots n_k$. (E.g Number of ways n people can be arranged in a line $= n!$)

Division rule: $f : A \to B$ is a k-to-1 function, then, $|A| = k|B|$. (E.g. For arranging people around a round table, f : seatings $\to$ arrangements is an n-to-1 function).

Counting subsets (Combinations)

Q: How many size-k subsets of a size-n set are there?
Example: How many ways can we select 5 books from 100?

Let us form a permutation of the n elements, and pick the first k elements to form the subset. Every size k subset can be generated this way. There are $n!$ total such permutations.

The order of the first k elements in the permutation does not matter in forming the subset, and neither does the order of the remaining $n-k$ elements.

The first k elements can be ordered in $k!$ ways and the remaining $n-k$ elements can be ordered in $(n-k)!$ ways. Using the product rule, $k! \times (n-k)!$ permutations map to the same size k subset.

Hence, the function f : permutations $\to$ size k subsets is a $k! \times (n-k)!$-to-1 function. By the division rule, $|\text{permutations}| = k! \times (n-k)!\, |\text{size } k \text{ subsets}|$. Hence, the total number of size k subsets $= \frac{n!}{k! \times (n-k)!}$.

n **choose** $k = \binom{n}{k} := \frac{n!}{k! \times (n-k)!}$.

Counting subsets (Combinations)

Q: Prove that $\binom{n}{k} = \binom{n}{n-k}$ - both algebraically (using the formula for $\binom{n}{k}$) and combinatorially (without using the formula)

Ans: Algebraically, $\binom{n}{k}$ is symmetric with respect to k and $n-k$. Combinatorially, number of ways of choosing elements to form a set of size k = number of ways of choosing $n-k$ elements to discard.

Q: Which is bigger? $\binom{8}{4}$ vs $\binom{8}{5}$? Ans: $\binom{8}{4} = 70$. $\binom{8}{5} = 56$

Counting subsets – Example

Q: How many m-bit binary sequences contain exactly k ones?

Consider set $A = \{1, \ldots, m\}$ and selecting S, a subset of size k. For example, say $m = 10, k = 3$ and $S = \{3, 7, 10\}$. S records the positions of the 1's, and can mapped to the sequence 001 0001 001. Similarly, every m-bit sequence with exactly k ones can be mapped to a subset S of size k. Hence, there is a bijection:
f : m-bit sequence with exactly k ones $\rightarrow$ subsets of size k from size m-set, and
$|m$-bit sequence with exactly k ones$| = |$subsets of size $k| = \binom{m}{k}$.

Q: Suppose we want to buy 10 donuts. There are 5 donut varieties – chocolate, lemon-filled, sugar, glazed, plain. What is the number of ways to select the 10 donuts?

Recall that the number of ways of selecting 10 donuts with 5 varieties = number of 14-bit sequences with exactly 4 ones $= \binom{14}{4} = 1001$.

Q: What is the number of ways of choosing n things with k varieties?

Ans: Equal to the number of $n + k - 1$-bit sequences with exactly $k - 1$ ones $= \binom{n+k-1}{k-1}$.

Counting subsets – Example

Q: What is the number of n-bit binary sequences with at least k ones?

Ans: Set of n-bit binary sequences with at least k ones $=$ n-bit binary sequences with exactly k ones $\cup$ n-bit binary sequences with exactly $k+1$ ones $\cup \ldots \cup$ n-bit binary sequences with exactly n ones. By the sum rule for disjoint sets, number of n-bit binary sequences with at least k ones $= \sum_{i=k}^{n} \binom{n}{i}$.

Q: What is the number of n-bit binary sequences with less than k ones?

Ans: $\sum_{i=0}^{k-1} \binom{n}{i}$

Q: What is the total number of n-bit binary sequences?

Ans: 2^n

Total number of n-bit binary sequences $=$ number of n-bit binary sequences with at least k ones $+$ number of n-bit binary sequences with less than k ones.
Combining the above answers, we can conclude that, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Have recovered a special case of the binomial theorem!

Binomial Theorem

For all $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Example: If $a = b = 1$, then $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ (result from previous slide).

If $n = 2$, then $(a+b)^2 = \binom{2}{0} a^2 + \binom{2}{1} ab + \binom{2}{2} b^2 = a^2 + 2ab + b^2$.

Q: What is the coefficient of the terms with ab^3 and $a^2 b^3$ in $(a+b)^4$? Ans: $\binom{4}{1} = \binom{4}{3}$, 0.

Q: For $a, b > 0$, what is the coefficient of $a^{2n-7} b^7$ and $a^{2n-8} b^8$ in $(a+b)^{2n} + (a-b)^{2n}$?

Ans: $(a+b)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} a^{2n-k} b^k$,
$(a-b)^{2n} = -\sum_{k=0}^{2n} \binom{2n}{k} a^{2n-k} b^k \mathcal{I}\{\text{k is odd}\} + \sum_{k=0}^{2n} \binom{2n}{k} a^{2n-k} b^k \mathcal{I}\{\text{k is even}\}$.
$(a+b)^{2n} + (a-b)^{2n} = 2\sum_{k=0}^{2n} \binom{2n}{k} a^{2n-k} b^k \mathcal{I}\{\text{k is even}\}$. Hence, coefficient of $a^{2n-7} b^7 = 0$, coefficient of $a^{2n-8} b^8 = 2\binom{2n}{8}$.

Counting Practice

Q: A standard dice (with numbers $\{1,2,3,4,5,6\}$) is rolled 6 times in succession.

- How many rolls will have no 6? Ans: 5^6. Since each of the 6 positions (corresponding to a roll) can be filled in 5 ways (all except 6).
- How many rolls will have each number once? Ans: $6!$. The first position can be filled in 6 ways. Since we need each number to come up only once, the second position can be filled in 5 ways, and so on.
- How many rolls will have 6 come up exactly once? Ans: In order to count such rolls, first pick the roll number where the 6 appears. This can be done in 6 ways. For each choice, the remaining 5 positions (corresponding to the rolls) can be filled in 5 ways. Hence, the total number of ways $= 6 \times 5^5$
- How many rolls will have 6 come up exactly k times (for $k \leq 6$)? Ans: Similar to the above question, first pick the k rolls where a 6 appears, and the remaining $6-k$ positions can be filled in 5 ways. Hence, the total number of ways $= \binom{6}{k} \times 5^{6-k}$

Counting Practice

Q: How many 5 digit numbers are there which contain at least one zero? Note that a number is different from a string, i.e. 01234 is not a 5-digit number and is hence not allowed.

Ans: We need to first choose the number of zeros and then fill out the remaining digits. The number of zeros can be $\{1, 2, 3, 4\}$ (since 00000 is not a 5 digit number). Let D_i be the set of numbers that contain i zeros. Hence, the total number of numbers is $|D_1 \cup D_2 \cup D_3 \cup D_4| = \sum_{i=1}^{4} |D_i|$ (using the sum rule since the sets are disjoint). $|D_i| = \binom{4}{i} \times 9^{5-i}$ corresponding to choosing i positions for the zero (the first position cannot be chosen) and filling the remaining positions by one of the remaining 9 digits. Hence, the total number of ways is equal to $\sum_{i=1}^{4} \binom{4}{i} \times 9^{5-i}$.

Counting Practice

Q: How many non-negative integer solutions ($x_1, x_2, x_3 \geq 0$) are there to the following equation:

$$x_1 + x_2 + x_3 = 40$$

Ans: This is similar to the donuts question where there are 3 donuts varieties – x_1, x_2 and x_3, and we need to choose 40 donuts. Using the earlier results, the number of possible ways this can be done is equal to the number of strings with length 42 with exactly 2 ones, which is equal to $\binom{42}{2}$.

Questions?