CMPT 210: Probability and Computing

Lecture 23

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Comparing the Bounds

For r.v's $T_1, T_2, \ldots T_n$, if $T_i \in \{0,1\}$ and $\Pr[T_i = 1] = p_i$. Define $T := \sum_{i=1}^n T_i$. By linearity of expectation, $\mathbb{E}[T] = \sum_{i=1}^n p_i$. For $c \geq 1$,

Markov's Theorem: $\Pr[T \geq c\mathbb{E}[T]] \leq \frac{1}{c}$. Does not require T_i's to be independent.

Chebyshev's Theorem:

$$\Pr[T - \mathbb{E}[T] \geq x] \leq \Pr[|T - \mathbb{E}[T]| \geq x] \leq \frac{\mathrm{Var}[T]}{x^2}$$

$$\implies \Pr[T - \mathbb{E}[T] \geq (c-1)\mathbb{E}[T]] \leq \frac{\mathrm{Var}[T]}{(c-1)^2\,(\mathbb{E}[T])^2)} \qquad (x = (c-1)\,\mathbb{E}[T])$$

If the T_i's are pairwise independent, by linearity of variance, $\mathrm{Var}[T] = \sum_{i=1}^n p_i\,(1-p_i)$. Hence, $\Pr[T \geq c\mathbb{E}[T]] \leq \frac{\sum_{i=1}^n p_i\,(1-p_i)}{(c-1)^2\left(\sum_{i=1}^n p_i\right)^2}$. If for all i, $p_i = 1/2$, then, $\Pr[T \geq c\mathbb{E}[T]] \leq \frac{1}{(c-1)^2\,n}$.

Chernoff Bound: If T_i' are mutually independent, then, $\Pr[T \geq c\mathbb{E}[T]] \leq \exp(-\beta(c)\,\mathbb{E}[T]) = \exp\left(-(c\ln(c) - c + 1)\,\left(\sum_{i=1}^n p_i\right)\right)$. If for all i, $p_i = 1/2$, $\Pr[T \geq c\mathbb{E}[T]] \leq \exp\left(-\frac{n\,(c\ln(c)-c+1)}{2}\right)$.

Chernoff Bound – Lottery Game

Q: Pick-4 is a lottery game in which you pay \$1 to pick a 4-digit number between 0000 and 9999. If your number comes up in a random drawing, then you win \$5,000. Your chance of winning is 1 in 10000. If 10 million people play, then the expected number of winners is 1000. When there are 1000 winners, the lottery keeps \$5 million of the \$10 million paid for tickets. The lottery operator's nightmare is that the number of winners is much greater – especially at the point where more than 2000 win and the lottery must pay out more than it received. What is the probability that will happen? (Assume that the players' picks and the winning number are random, independent and uniform)

Let T_i be an indicator for the event that player i wins. Then $T := \sum_{i=1}^{n} T_i$ is the total number of winners. Using the independence assumptions, we can conclude that T_i are independent, as required by the Chernoff bound.

We wish to compute $\Pr[T \geq 2000] = \Pr[T \geq 2\mathbb{E}[T]]$. Hence $c = 2$ and $\beta(c) \approx 0.386$. By the Chernoff bound,

$$\Pr[T \geq 2\mathbb{E}[T]] \leq \exp(-\beta(c)\mathbb{E}[T]) = \exp(-(0.386)\,1000) < \exp(-386) \approx 10^{-168}$$

Questions?

Chernoff Bound – Proof

Chernoff Bound: Let $T_1, T_2, \ldots, T_n$ be mutually independent r.v's such that $0 \leq T_i \leq 1$ for all i. If $T := \sum_{i=1}^{n} T_i$, for all $c \geq 1$ and $\beta(c) := c\ln(c) - c + 1$,

$$\Pr[T \geq c\mathbb{E}[T]] \leq \exp(-\beta(c)\,\mathbb{E}[T])$$

Proof: We want to compute $\Pr[T \geq c\mathbb{E}[T]] = \Pr[f(T) \geq f(c\mathbb{E}[T])]$ where f is a one-one monotonically non-decreasing function. For $c \geq 1$, choosing $f(T) = c^T$ and using Markov's Theorem,

$$\Pr[T \geq c\mathbb{E}[T]] = \Pr[c^T \geq c^{c\mathbb{E}[T]}] \leq \frac{\mathbb{E}[c^T]}{c^{c\mathbb{E}[T]}}$$

$$\leq \frac{\exp((c-1)\,\mathbb{E}[T])}{c^{c\mathbb{E}[T]}} \qquad \text{(To prove next: } \mathbb{E}[c^T] \leq \exp((c-1)\,\mathbb{E}[T])\text{)}$$

$$= \frac{\exp((c-1)\,\mathbb{E}[T])}{\exp(\ln(c^{c\mathbb{E}[T]}))} = \frac{\exp((c-1)\,\mathbb{E}[T])}{\exp(c\mathbb{E}[T]\,\ln(c))} = \exp\left(-(c\ln(c) - c + 1)\,\mathbb{E}[T]\right)$$

$$\implies \Pr[T \geq c\mathbb{E}[T]] \leq \exp\left(-\beta(c)\,\mathbb{E}[T]\right)$$

The proof would be done if we prove that $\mathbb{E}[c^T] \leq \exp((c-1)\,\mathbb{E}[T])$ and we do this next.

Chernoff Bound – Proof

Claim: $\mathbb{E}[c^T] \leq \exp((c-1)\,\mathbb{E}[T])$

$$\mathbb{E}[c^T] = \mathbb{E}[c^{\sum_{i=1}^{n} T_i}] = \mathbb{E}\left[\prod_{i=1}^{n} c^{T_i}\right] = \prod_{i=1}^{n} \mathbb{E}[c^{T_i}]$$

(Expectation of product of mutually independent r.v's is equal to the product of the expectation.)

$$\leq \prod_{i=1}^{n} \exp((c-1)\,\mathbb{E}[T_i]) \qquad \text{(To prove next: } \mathbb{E}[c^{T_i}] \leq \exp((c-1)\,\mathbb{E}[T_i])\text{)}$$

$$= \exp\left((c-1)\sum_{i=1}^{n} \mathbb{E}[T_i]\right) = \exp\left((c-1)\,\mathbb{E}\left[\sum_{i=1}^{n} T_i\right]\right)$$

(Linearity of Expectation)

$$\implies \mathbb{E}[c^T] \leq \exp((c-1)\mathbb{E}[T])$$

The proof would be done if we prove that $\mathbb{E}[c^{T_i}] \leq \exp((c-1)\,\mathbb{E}[T_i])$ and we do this next.

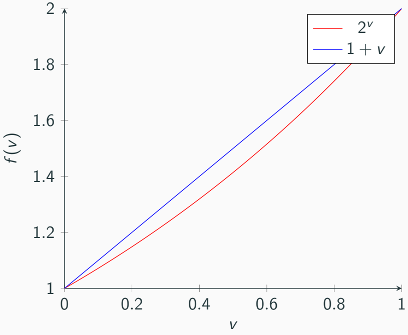
Chernoff Bound – Proof

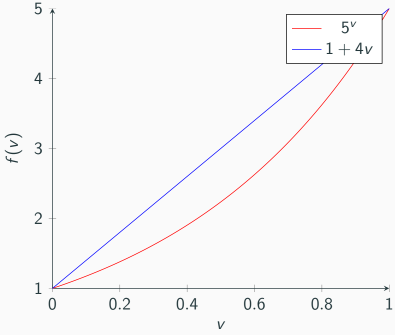
Claim: $\mathbb{E}[c^{T_i}] \leq \exp((c-1)\,\mathbb{E}[T_i])$

$$\mathbb{E}[c^{T_i}] = \sum_{v \in \mathrm{Range}(T_i)} c^v \Pr[T_i = v] \leq \sum_{v \in \mathrm{Range}(T_i)} (1 + (c-1)v)\ \Pr[T_i = v]$$

(Since $T_i \in [0,1]$ and $c^v \leq 1 + (c-1)v$ for all $v \in [0,1]$.)

For $c = 2$ and $c = 5$,

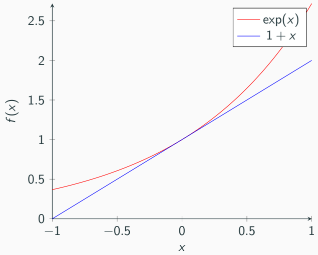




Chernoff Bound – Proof

$$
\begin{aligned}
\mathbb{E}[c^{T_i}] &\le \sum_{v \in \mathrm{Range}(T_i)} (1 + (c-1)v) \Pr[T_i = v] \\
&= \sum_{v \in \mathrm{Range}(T_i)} \Pr[T_i = v] + (c-1) \sum_{v \in \mathrm{Range}(T_i)} v \Pr[T_i = v] \\
&= 1 + (c-1)\,\mathbb{E}[T_i] \le \exp((c-1)\mathbb{E}[T_i]) \qquad (\text{Since } 1 + x \le \exp(x) \text{ for all } x)
\end{aligned}
$$

$$
\implies \mathbb{E}[c^{T_i}] \le \exp((c-1)\,\mathbb{E}[T_i])
$$



Hence we have proved the Chernoff Bound!

Questions?