CMPT 210: Probability and Computing

Lecture 18

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Joint distributions

For a given experiment, we are often interested not only in the PDFs of individual random variables but also in the relationships between two or more random variables. For example, we might be interested in the mean time of failure and its connection with different number of components in the system.

A joint distribution between r.v's X and Y can be specified by its joint PDF as follows:

$$\mathrm{PDF}_{X,Y}[x,y] = \Pr[X = x \cap Y = y]$$

If X and Y are independent random variables, $\mathrm{PDF}_{X,Y}[x,y] = \mathrm{PDF}_X[x]\,\mathrm{PDF}_Y[y]$.

If $\mathrm{Range}[X] = \{x_1, x_2, \ldots x_n\}$, $\mathrm{Range}[Y] = \{y_1, y_2, \ldots y_n\}$, then for $x \in \mathrm{Range}(X)$,
$[X = x] = [X = x \cap y = y_1] \cup [X = x \cap y = y_2] \cup \ldots \cup [X = x \cap y = y_n]$
$\implies \Pr[X = x] = \Pr[X = x \cap y = y_1] + \Pr[X = x \cap y = y_2] + \ldots + \Pr[X = x \cap y = y_n]$.

$\implies \mathrm{PDF}_X[x] = \sum_i \mathrm{PDF}_{X,Y}[x, y_i]$.
Hence, we can obtain the distribution for each r.v. from the joint distribution by "marginalizing" over the other r.v's.

Joint distributions - Examples

Q: Suppose that 3 batteries are randomly chosen from a group of 3 new, 4 used but still working, and 5 defective batteries. If the batteries are distinct and we let X and Y denote, respectively, the number of new and used but still working batteries that are chosen, completely specify $\text{PDF}_{X,Y}$.

For $i \in [3], j \in [3]$, $\text{PDF}_{X,Y}[i,j] = \Pr[X = i \cap Y = j | X + Y \leq 3] = \frac{\binom{3}{i}\binom{4}{j}\binom{5}{3-i-j}}{\binom{12}{3}}$.

$\text{PDF}_{X,Y}[0,0] = \frac{\binom{5}{3}}{\binom{12}{3}} = 10/220$, $\text{PDF}_{X,Y}[1,2] = \frac{\binom{3}{1}\binom{4}{2}\binom{5}{0}}{\binom{12}{3}} = 18/220$.

Table 4.1 $P\{X = i, Y = j\}$.

i \ j	0	1	2	3	Row Sum $= P\{X = i\}$
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$\frac{4}{220}$	$\frac{84}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0	$\frac{108}{220}$
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0	$\frac{27}{220}$
3	$\frac{1}{220}$	0	0	0	$\frac{1}{220}$
Column Sums $= P\{Y = j\}$	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$	

Questions?

Expectation - Examples

For a random variable $X : \mathcal{S} \to V$ and a function $g : V \to \mathbb{R}$, we define $\mathbb{E}[g(X)]$ as follows:

$$\mathbb{E}[g(X)] := \sum_{x \in \text{Range}(X)} g(x) \Pr[X = x]$$

If $g(x) = x$ for all $x \in \text{Range}(X)$, then $\mathbb{E}[g(X)] = \mathbb{E}[X]$.

Q: For a standard dice, if X is the r.v. corresponding to the number that comes up on the dice, compute $\mathbb{E}[X^2]$ and $(\mathbb{E}[X])^2$

For a standard dice, $X \sim \text{Uniform}(\{1, 2, 3, 4, 5, 6\})$ and hence,

$$\mathbb{E}[X^2] = \sum_{x \in \{1,2,3,4,5,6\}} x^2 \Pr[X = x] = \frac{1}{6}\left[1^2 + 2^2 + \ldots + 6^2\right] = \frac{91}{6}$$

$$(\mathbb{E}[X])^2 = \left(\sum_{x \in \{1,2,3,4,5,6\}} x \Pr[X = x]\right)^2 = \left(\frac{1}{6}\left[1 + 2 + \ldots + 6\right]\right)^2 = \frac{49}{4}$$

Deviation from the Mean

We have developed tools to calculate the mean of random variables. Getting a handle on the expectation is useful because it tell us what would happen on average.

Summarizing the PDF using the mean is typically not enough. We also want to know how "spread" the distribution is.

Example: Consider three random variables W, Y and Z whose PDF's can be given as:

$$\begin{aligned} W &= 0 && \text{(with } p = 1) \\ Y &= -1 && \text{(with } p = 1/2) \\ &= +1 && \text{(with } p = 1/2) \\ Z &= -1000 && \text{(with } p = 1/2) \\ &= +1000 && \text{(with } p = 1/2) \end{aligned}$$

Though $\mathbb{E}[W] = \mathbb{E}[Y] = \mathbb{E}[Z] = 0$, these distributions are quite different. Z can take values really far away from its expected value, while W can take only one value equal to the mean. Hence, we want to understand how much does a random variable "deviate" from its mean.

Variance

Standard way to measure the deviation from the mean is to calculate the *variance*. For r.v. X,

$$\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_{x \in \text{Range}(X)} (x - \mu)^2 \Pr[X = x] \qquad (\text{where } \mu := \mathbb{E}[X])$$

Intuitively, the variance measures the weighted (by the probability) average of how far (in squared distance) the random variable is from the mean μ.

Q: If $X \sim \text{Ber}(p)$, compute $\text{Var}[X]$.

Since X is a Bernoulli random variable, $X = 1$ with probability p and $X = 0$ with probability $1 - p$. Recall that $\mathbb{E}[X] = \mu = (0)(1-p) + (1)(p) = p$.

$$\text{Var}[X] = \sum_{x \in \{0,1\}} (x - p)^2 \Pr[X = x] = (0 - p)^2 \Pr[X = 0] + (1 - p)^2 \Pr[X = 1]$$

$$= p^2(1-p) + (1-p)^2 p = p(1-p)[p + 1 - p] = p(1-p).$$

For a Bernoulli r.v. X, $\text{Var}[X] = p(1-p) \leq \frac{1}{4}$. Hence, the variance is maximum when $p = 1/2$ (equal probability of getting heads/tails).

Variance

Alternate definition of variance: $\mathrm{Var}[X] = \mathbb{E}[X^2] - \mu^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.

$$\begin{aligned}
\textit{Proof}: \mathrm{Var}[X] &= \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_{x \in \mathrm{Range}(X)} (x - \mu)^2 \Pr[X = x] \\
&= \sum_{x \in \mathrm{Range}(X)} (x^2 - 2\mu x + \mu^2) \Pr[X = x] \\
&= \sum_{x \in \mathrm{Range}(X)} (x^2 \Pr[X = x]) - (2\mu x \Pr[X = x]) + (\mu^2) \Pr[X = x] \\
&= \sum_{x \in \mathrm{Range}(X)} x^2 \Pr[X = x] - 2\mu \sum_{x \in \mathrm{Range}(X)} x \Pr[X = x] + \mu^2 \sum_{x \in \mathrm{Range}(X)} \Pr[X = x] \\
&\qquad \text{(Since } \mu \text{ is a constant does not depend on the } x \text{ in the sum.)} \\
&= \mathbb{E}[X^2] - 2\mu \mathbb{E}[X] + \mu^2 \sum_{x \in \mathrm{Range}(X)} \Pr[X = x] \quad \text{(Definition of } \mathbb{E}[X] \text{ and } \mathbb{E}[X^2]) \\
&= \mathbb{E}[X^2] - 2\mu^2 + \mu^2 \qquad \text{(Definition of } \mu) \\
\implies \mathrm{Var}[X] &= \mathbb{E}[X^2] - \mu^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.
\end{aligned}$$

Back to throwing dice

Q: For a standard dice, if X is the r.v. equal to the number that comes up, compute $\text{Var}[X]$.

Recall that, for a standard dice, $X \sim \text{Uniform}(\{1,2,3,4,5,6\})$ and hence,

$$\mathbb{E}[X^2] = \sum_{x \in \{1,2,3,4,5,6\}} x^2 \Pr[X = x] = \frac{1}{6}\left[1^2 + 2^2 + \ldots + 6^2\right] = \frac{91}{6}$$

$$(\mathbb{E}[X])^2 = \left(\sum_{x \in \{1,2,3,4,5,6\}} x \Pr[X = x]\right)^2 = \left(\frac{1}{6}\left[1 + 2 + \ldots + 6\right]\right)^2 = \frac{49}{4}$$

$$\implies \text{Var}[X] = \frac{91}{6} - \frac{49}{4} \approx 2.917$$

Q: If $X \sim \text{Uniform}(\{v_1, v_2, \ldots v_n\})$, compute $\text{Var}[X]$.

$$\mathbb{E}[X] = \sum_{i=1}^{n} v_i \Pr[X = v_i] = \frac{1}{n}\left[v_1 + v_2 + \ldots v_n\right] \quad ; \quad \mathbb{E}[X^2] = \frac{1}{n}\left[v_1^2 + v_2^2 + \ldots v_n^2\right].$$

$$\implies \text{Var}[X] = \frac{\left[v_1^2 + v_2^2 + \ldots v_n^2\right]}{n} - \left(\frac{\left[v_1 + v_2 + \ldots v_n\right]}{n}\right)^2$$

Variance - Examples

Q: Calculate Var[W], Var[Y] and Var[Z] whose PDF's are given as:

$$W = 0 \quad \text{(with } p = 1\text{)}$$
$$Y = -1 \quad \text{(with } p = 1/2\text{)}$$
$$= +1 \quad \text{(with } p = 1/2\text{)}$$
$$Z = -1000 \quad \text{(with } p = 1/2\text{)}$$
$$= +1000 \quad \text{(with } p = 1/2\text{)}$$

Recall that $\mathbb{E}[W] = \mathbb{E}[Y] = \mathbb{E}[Z] = 0$.

$\text{Var}[W] = \mathbb{E}[W^2] - (\mathbb{E}[W])^2 = \mathbb{E}[W^2] = \sum_{w \in \text{Range}(W)} w^2 \Pr[W = w] = 0^2(1) = 0$. The variance of W is zero because it can only take one value and the r.v. does not "vary".

$\text{Var}[Y] = \mathbb{E}[Y^2] = \sum_{y \in \text{Range}(Y)} y^2 \Pr[Y = y] = (-1)^2(1/2) + (1)^2(1/2) = 1$.

$\text{Var}[Z] = \mathbb{E}[Z^2] = \sum_{z \in \text{Range}(Z)} z^2 \Pr[Z = z] = (-1000)^2(1/2) + (1000)^2(1/2) = 10^6$.

Hence, the variance can be used to distinguish between r.v.'s that have the same mean.