CMPT 210: Probability and Computing

Lecture 17

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Recap

Expectation/mean of a random variable R is denoted by $\mathbb{E}[R]$ and "summarizes" its distribution. Formally, $\mathbb{E}[R] := \sum_{\omega \in \mathcal{S}} \Pr[\omega]\, R[\omega]$

Alternate definition of expectation: $\mathbb{E}[R] = \sum_{x \in \text{Range}(R)} x \, \Pr[R = x]$.

Linearity of Expectation: For n random variables $R_1, R_2, \ldots, R_n$ and constants $a_1, a_2, \ldots, a_n$, $\mathbb{E}\left[\sum_{i=1}^{n} a_i R_i\right] = \sum_{i=1}^{n} a_i \, \mathbb{E}[R_i]$.

Conditional Expectation: For random variable R, the expected value of R conditioned on an event A is given by:

$$\mathbb{E}[R|A] = \sum_{x \in \text{Range}(R)} x \, \Pr[R = x|A]$$

Law of Total Expectation

If R is a random variable $\mathcal{S} \to V$ and events $A_1, A_2, \ldots A_n$ form a partition of the sample space i.e. for all i, j, $A_i \cap A_j = \emptyset$ and $A_1 \cup A_2 \cup \ldots \cup A_n = \mathcal{S}$, then,

$$\mathbb{E}[R] = \sum_i \mathbb{E}[R|A_i] \Pr[A_i] \,.$$

Proof:

$$\mathbb{E}[R] = \sum_{x \in \text{Range}(R)} x \Pr[R = x] = \sum_{x \in \text{Range}(R)} x \sum_i \Pr[R = x | A_i] \Pr[A_i]$$

(Law of total probability)

$$= \sum_i \Pr[A_i] \sum_{x \in \text{Range}(R)} x \Pr[R = x | A_i]$$

$$\implies \mathbb{E}[R] = \sum_i \Pr[A_i] \, \mathbb{E}[R|A_i].$$

Conditional Expectation - Examples

Q: Suppose that 49.6% of the people in the world are male and the rest female. If the expected height of a randomly chosen male is 5 feet 11 inches, while the expected height of a randomly chosen female is 5 feet 5 inches, what is the expected height of a randomly chosen person?

Define H to be the random variable equal to the height (in feet) of a randomly chosen person.
Define M to be the event that the person is male and F the event that the person is female.
We wish to compute $\mathbb{E}[H]$ and we know that $\mathbb{E}[H|M] = 5 + \frac{11}{12}$ and $\mathbb{E}[H|F] = 5 + \frac{5}{12}$.
$\Pr[M] = 0.496$ and $\Pr[F] = 1 - 0.496 = 0.504$.
Hence, $\mathbb{E}[H] = \mathbb{E}[H|M] \Pr[M] + \mathbb{E}[H|F] \Pr[F] = \frac{71}{12}(0.496) + \frac{65}{12}(0.504)$.

Questions?

Randomized Quick Select

Given an array A of n distinct numbers, return the k^{th} smallest element in A for $k \in [1, n]$.

Algorithm Randomized Quick Select

```
function QuickSelect(A, k)
If Length(A) = 1, return A[1].
Select p ∈ A uniformly at random.
Construct sets Left := {x ∈ A | x < p} and Right := {x ∈ A | x > p}.
r = |Left| + 1 {Element p is the r^th smallest element in A.}
if k = r then
    return p
else if k < r then
    QuickSelect(Left, k)
else
    QuickSelect(Right, k − r)
end if
```

Randomized Quick Select

If $A = \{2, 7, 0, 1, 3\}$ and we wish to find the 2^{nd} smallest element meaning that $k = 2$. According to the algorithm, $p \sim \text{Uniform}(A)$. Say $p = 3$.

Then after step 1, $\text{Left} = \{2, 0, 1\}$ and $\text{Right} = \{7\}$. $r := |\text{Left}| + 1 = 3 + 1 = 4$. Since $r > k$, we recurse on the left-hand side by calling the algorithm on $\{2, 0, 1\}$ with $k = 2$.

$p \sim \text{Uniform}(\{2, 0, 1\})$. Say $p = 1$. After step 2, $\text{Left} = \{0\}$ and $\text{Right} = \{2\}$. $r := |\text{Left}| + 1 = 1 + 1 = 2$. Since $r = k$, we terminate the recursion and return $p = 1$ as the second-smallest element in A.

Q: Run the algorithm if $p = 0$ in the first step? Ans: $\text{Left} = \{\}$ and $\text{Right} = \{2, 7, 1, 3\}$. Hence $r = 1 < k = 2$. Hence we will recurse on the right-hand side by calling the algorithm on $\{2, 7, 1, 3\}$ with $k = 1$.

Q: Run the algorithm if $p = 1$ in the first step? Ans: $\text{Left} = \{0\}$ and $\text{Right} = \{2, 7, 3\}$. Hence $r = 1 + 1 = 2$. Hence we will return the pivot element $p = 1$.

Randomized Quick Select – Analysis

Alternate way: Sort the elements in A and return the k^{th} element in the sorted list. Uses $O(n \log(n))$ comparisons.

Q: Can Randomized Quick Select do better – what is the maximum number of comparisons required by Randomized Quick Select in the worst-case? Ans: $O(n^2)$ when $k = n$ and the pivots are chosen in increasing order.

In the worst case, Randomized Quick Select is worse than the naive strategy of sorting and returning the k^{th} element. What about the average (over the pivot selection) case?

Claim: For any array A with n distinct elements, and for any $k \in [n]$, Randomized Quick Select performs fewer than $8n$ comparisons in expectation.

In order to prove this claim, we will need to prove the following lemma.

Randomized Quick Select – Analysis

Lemma: The child sub-problem's array (either Left or Right) after the partitioning (in Line 4 of the algorithm) has expected size smaller than $\frac{7n}{8}$.

Proof: Define a "good" event $\mathcal{E}$ that the randomly chosen pivot splits the array roughly in half.

Formally, if n is the length of the array, then $\mathcal{E}$ is the event that $r \in \left(\frac{n}{4}, \frac{3n}{4}\right]$ (for simplicity, let us assume that n is divisible by 4.) Since p is chosen uniformly at random, $\Pr[\mathcal{E}] = \frac{3n/4 - n/4}{n} = \frac{1}{2}$.

Recall that $|\text{Left}| = r - 1$ and $|\text{Right}| = n - r$. Hence if event $\mathcal{E}$ happens, then $|\text{Left}| < \frac{3n}{4}$ and $|\text{Right}| < \frac{3n}{4}$. Hence, $|\text{Child}| < \frac{3n}{4}$. If event $\mathcal{E}$ does not happen, in the worst-case, $|\text{Child}| < n$. By using the law of total expectation,

$$\begin{aligned}\mathbb{E}[|\text{Child}|] &= \mathbb{E}[|\text{Child}| \,|\mathcal{E}] \Pr[\mathcal{E}] + \mathbb{E}[|\text{Child}| \,|\mathcal{E}^c] \Pr[\mathcal{E}^c] \\ &< \frac{3n}{4}\frac{1}{2} + (n)\frac{1}{2} = \frac{7n}{8}.\end{aligned}$$

Hence on average, the size of the child sub-problem is smaller than $\frac{7n}{8}$, proving the lemma.

Randomized Quick Select – Analysis

In order to upper-bound the total number of comparisons, we use the Lemma with an induction on n. Recall that we need to prove that Randomized Quick Select requires fewer than $8n$ comparisons in expectation.

Base case: If $n = 1$, then we require $0 < 8(1)$ comparisons. Hence the base case is satisfied.

Inductive Step: Assume that for all $m < n$,
$\mathbb{E}[\text{Total number of comparisons for size } m \text{ array}] < 8\,m$.

$$
\begin{aligned}
&\mathbb{E}[\text{Total number of comparisons for size } n \text{ array}] \\
&= \mathbb{E}[(n-1) + \text{Total number of comparisons in child sub-problem}] \\
&= (n-1) + \mathbb{E}[\text{Total number of comparisons in child sub-problem}] && \text{(Linearity of expectation)} \\
&< (n-1) + 8\,\mathbb{E}[|\text{Child}|] && \text{(Induction hypothesis)} \\
&< (n-1) + 8\frac{7n}{8} < 8n. && \text{(Lemma)}
\end{aligned}
$$

Hence, for any $k \in [n]$, on average, Randomized Quick Select requires fewer than $8n$ comparisons, even though it might require $O(n^2)$ comparisons in the worst-case.

Questions?

We define two random variables R_1 and R_2 to be independent if for *all* $x_1 \in \text{Range}(R_1)$ and $x_2 \in \text{Range}(R_2)$, events $[R_1 = x_1]$ and $[R_2 = x_2]$ are independent. More formally, we require,

$$\Pr[(R_1 = x_1) \cap (R_2 = x_2)] = \Pr[(R_1 = x_1)] \Pr[(R_2 = x_2)]$$

Q: Suppose we toss three independent, unbiased coins. Let C be r.v. equal to the number of heads that appear and M be the r.v. that is equal to 1 if all the coins match. Are random variables C and M independent?

$\text{Range}(C) = \{0, 1, 2, 3\}$ and $\text{Range}(M) = \{0, 1\}$. $\Pr[C = 3] = \frac{1}{8}$ and $\Pr[M = 1] = \frac{1}{4}$. $\Pr[(C = 3) \cap (M = 1)] = \frac{1}{8} \neq \frac{1}{32} = \Pr[C = 3] \Pr[M = 1]$. Hence, C and M are not independent.

Independence - Examples

Q: If H_1 is the indicator r.v. equal to one if the first toss is a heads, are H_1 and M independent?

$\Pr[H_1 = 1] = \Pr[H_1 = 0] = \frac{1}{2}$, $\Pr[M = 1] = \frac{1}{4}$, $\Pr[M = 0] = \frac{3}{4}$.

$\Pr[H_1 = 0 \cap M = 1] = \Pr[\{TTT\}] = \frac{1}{8} = \Pr[H_1 = 0] \Pr[M = 1]$.

$\Pr[H_1 = 1 \cap M = 1] = \Pr[\{HHH\}] = \frac{1}{8} = \Pr[H_1 = 1] \Pr[M = 1]$.

$\Pr[H_1 = 0 \cap M = 0] = \Pr[\{THH, THT, TTH\}] = \frac{3}{8} = \Pr[H_1 = 0] \Pr[M = 0]$.

$\Pr[H_1 = 1 \cap M = 0] = \Pr[\{HHT, HTH, HTT\}] = \frac{3}{8} = \Pr[H_1 = 1] \Pr[M = 0]$.

Hence, H_1 and M are independent.

Independence of random variables

Q: If R_1 and R_2 are not independent, is $\mathbb{E}[R_1 + R_2] = \mathbb{E}[R_1] + \mathbb{E}[R_2]$?

Yes! Recall the derivation of the linearity of expectation. We never assumed that R_1 and R_2 are independent for the proof and the linearity of expectation holds regardless of whether the random variables are independent.

Q: If R_1 and R_2 are independent, is $\mathbb{E}[R_1 R_2] = \mathbb{E}[R_1]\,\mathbb{E}[R_2]$? Yes!

$$\mathbb{E}[R_1 R_2] = \sum_{x \in \text{Range}(R_1 R_2)} x \Pr[R_1 R_2 = x] = \sum_{r_1 \in \text{Range}(R_1), r_2 \in \text{Range}(R_2)} r_1 r_2 \Pr[R_1 = r_1 \cap R_2 = r_2] \qquad (x = r_1\, r_2)$$

$$= \sum_{r_1 \in \text{Range}(R_1)} \sum_{r_2 \in \text{Range}(R_2)} r_1 r_2 \Pr[R_1 = r_1 \cap R_2 = r_2] \qquad \text{(Splitting the sum)}$$

$$= \sum_{r_1 \in \text{Range}(R_1)} \sum_{r_2 \in \text{Range}(R_2)} r_1 r_2 \Pr[R_1 = r_1] \Pr[R_2 = r_2] \qquad \text{(Independence)}$$

$$= \sum_{r_1 \in \text{Range}(R_1)} r_1 \Pr[R_1 = r_1] \sum_{r_2 \in \text{Range}(R_2)} r_2 \Pr[R_2 = r_2] = \mathbb{E}[R_1]\mathbb{E}[R_2]$$

Independence of random variables

Alternate definition of independence – two random variables R_1 and R_2 are independent if for *all* $x_1 \in \text{Range}(R_1)$ and $x_2 \in \text{Range}(R_2)$,

$$\Pr[(R_1 = x_1)|(R_2 = x_2)] = \Pr[(R_1 = x_1)]$$
$$\Pr[(R_2 = x_2)|(R_1 = x_1)] = \Pr[(R_2 = x_2)]$$

Similar to events, random variables $R_1, R_2, \ldots, R_n$ are mutually independent if for all $x_1, x_2, \ldots, x_n$, events $[R_1 = x_1], [R_2 = x_2], \ldots [R_n = x_n]$ are mutually independent.

Mutual Independence of events: A set of events is said to be mutually independent if the probability of each event in the set is the same no matter which of the events has occurred. For events E_1, E_2 and E_3 to be mutually independent, all the following equalities should hold:

$$\Pr[E_1 \cap E_2] = \Pr[E_1]\Pr[E_2] \quad \Pr[E_1 \cap E_3] = \Pr[E_1]\Pr[E_3]$$
$$\Pr[E_2 \cap E_3] = \Pr[E_2]\Pr[E_3] \quad \Pr[E_1 \cap E_2 \cap E_3] = \Pr[E_1]\Pr[E_2]\Pr[E_3].$$

Alternatively, (i) $\forall i$ and $j \neq i$, $\Pr[E_i|E_j] = \Pr[E_i]$ and (ii) $\forall i$ and $j, k \neq i$, $\Pr[E_i|E_j \cap E_k] = \Pr[E_i]$.

Expectation/Independence - Examples

Q: Suppose there is a dinner party where n people check in their coats. The coats are mixed up during dinner, so that afterward each person receives a random coat. In particular, a person gets their own coat with probability $\frac{1}{n}$. What is the expected number of people who get their own coat?

Let G be the number of people that get their own coat. We wish to compute $\mathbb{E}[G]$. Define G_i to be the indicator r.v. that person i gets their own coat. Observe that $G = G_1 + G_2 + \ldots + G_n$ and by linearity of expectation $\mathbb{E}[G] = \mathbb{E}[G_1] + \mathbb{E}[G_2] + \ldots + \mathbb{E}[G_n]$. For each i, $\mathbb{E}[G_i] = \Pr[G_i] = \frac{1}{n}$. Hence, $\mathbb{E}[G] = 1$ meaning that on average one person will correctly receive their coat.

Q: If G_i is the indicator r.v. that person i gets their own coat, are the random variables $G_1, G_2, \ldots G_n$ mutually independent?

No. Since if $G_1 = G_2 = \ldots G_{n-1} = 1$, then,
$\Pr[G_n = 1 | (G_1 = 1 \cap G_2 = 1 \cap \ldots \cap G_{n-1} = 1)] = 1 \neq \frac{1}{n} = \Pr[G_n = 1]$. Conditioning on $(G_1, G_2, \ldots, G_{n-1})$ changes $\Pr[G_n]$, and hence the r.v's are not independent. Notice that we have used the linearity of expectation even though these r.v's are not mutually independent.

Questions?