CMPT 210: Probability and Computing

Lecture 10

Sharan Vaswani

February 8, 2024

Recap

Conditional probability: $\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$.

Multiplication Rule: For events $E_1, E_2, \ldots, E_n$,
$\Pr[E_1 \cap E_2 \ldots \cap E_n] = \Pr[E_1] \Pr[E_2|E_1] \Pr[E_3|E_1 \cap E_2] \ldots \Pr[E_n|E_1 \cap E_2 \cap \ldots E_{n-1}]$.

Conditional probability for complement events: For events E, F, $\Pr[E^c|F] = 1 - \Pr[E|F]$.

Bayes Rule: For events E and F if $\Pr[E] \neq 0$ and $\Pr[F] \neq 0$, then, $\Pr[F|E] = \frac{\Pr[E|F]\Pr[F]}{\Pr[E]}$.

Law of Total Probability and Bayes rule

Law of Total Probability: For events E and F, $\Pr[E] = \Pr[E|F]\Pr[F] + \Pr[E|F^c]\Pr[F^c]$.

Proof:

$$E = (E \cap F) \cup (E \cap F^c)$$

$$\implies \Pr[E] = \Pr[(E \cap F) \cup (E \cap F^c)] = \Pr[E \cap F] + \Pr[E \cap F^c]$$

(By union-rule for disjoint events)

$$\Pr[E] = \Pr[E|F]\Pr[F] + \Pr[E|F^c]\Pr[F^c] \qquad \text{(By definition of conditional probability)}$$

Combining Bayes rule and Law of total probability

$$\Pr[F|E] = \frac{\Pr[F \cap E]}{\Pr[E]} = \frac{\Pr[E|F]\,\Pr[F]}{\Pr[E]} \qquad \text{(By definition of conditional probability)}$$

$$\Pr[F|E] = \frac{\Pr[E|F]\,\Pr[F]}{\Pr[E|F]\Pr[F] + \Pr[E|F^c]\Pr[F^c]} \qquad \text{(By law of total probability)}$$

Total Probability - Examples

Q: In answering a question on a multiple-choice test, a student either knows the answer or she guesses. Let p be the probability that she knows the answer and $1 - p$ the probability that she guesses. Assume that a student who guesses at the answer will be correct with probability $\frac{1}{m}$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that she answered it correctly?

Let C be the event that the student answers the question correctly. Let K be the event that the student knows the answer. We wish to compute $\Pr[K|C]$.

We know that $\Pr[K] = p$ and $\Pr[C|K^c] = 1/m$, $\Pr[C|K] = 1$. Hence,
$\Pr[C] = \Pr[C|K]\Pr[K] + \Pr[C|K^c]\Pr[K^c] = (1)(p) + \frac{1}{m}(1 - p)$.

$\Pr[K|C] = \frac{\Pr[C|K]\ \Pr[K]}{\Pr[C]} = \frac{mp}{1+(m-1)p}$.

Total Probability - Examples

Q: An insurance company believes that people can be divided into two classes — those that are accident prone and those that are not. Their statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability 0.4, whereas this probability decreases to 0.2 for a non-accident-prone person. If we assume that 30% of the population is accident prone, what is the probability that a new policy holder will have an accident within a year of purchasing a policy?

Let A = event that a new policy holder will have an accident within a year of purchasing a policy. Let B = event that the new policy holder is accident prone. We know that $\Pr[B] = 0.3$, $\Pr[A|B] = 0.4$, $\Pr[A|B^c] = 0.2$. By the law of total probability,
$\Pr[A] = \Pr[A|B]\Pr[B] + \Pr[A|B^c]\Pr[B^c] = (0.4)(0.3) + (0.2)(0.7) = 0.26$.

Q: Suppose that a new policy holder has an accident within a year of purchasing their policy. What is the probability that they are accident prone?

Compute $\Pr[B|A] = \frac{\Pr[A|B]\ \Pr[B]}{\Pr[A]} = \frac{0.12}{0.26} = 0.4615$.

Total Probability - Examples

Q: Alice is taking a probability class and at the end of each week she can be either up-to-date or she may have fallen behind. If she is up-to-date in a given week, the probability that she will be up-to-date (or behind) in the next week is 0.8 (or 0.2, respectively). If she is behind in a given week, the probability that she will be up-to-date (or behind) in the next week is 0.6 (or 0.4, respectively). Alice is (by default) up-to-date when she starts the class. What is the probability that she is up-to-date after three weeks?

Let U_i and B_i be the events that Alice is up-to-date or behind respectively after i weeks. Since Alice starts the class up-to-date, $\Pr[U_1] = 0.8$ and $\Pr[B_1] = 0.2$. We also know that $\Pr[U_2|U_1] = 0.8$, $\Pr[U_3|U_2] = 0.8$ and $\Pr[B_2|U_1] = 0.2$, $\Pr[B_3|U_2] = 0.2$. Similarly, $\Pr[U_2|B_1] = 0.6$, $\Pr[U_3|B_2] = 0.6$ and $\Pr[B_2|B_1] = 0.4$, $\Pr[B_3|B_2] = 0.4$.

We wish to compute $\Pr[U_3]$. By the law of total probability,
$\Pr[U_3] = \Pr[U_3|U_2]\Pr[U_2] + \Pr[U_3|B_2]\Pr[B_2]$ and
$\Pr[U_2] = \Pr[U_2|U_1]\Pr[U_1] + \Pr[U_2|B_1]\Pr[B_1]$.

Hence, $\Pr[U_2] = (0.8)(0.8) + (0.6)(0.2) = 0.76$, and $\Pr[U_3] = (0.8)(0.76) + (0.6)(0.24) = 0.752$.

Simpson's Paradox

In 1973, there was a lawsuit against a university with the claim that a male candidate is more likely to be admitted to the university than a female.

Let us consider a simplified case – there are two departments, EE and CS, and men and women apply to the program of their choice. Let us define the following events: A is the event that the candidate is admitted to the program of their choice, F_E is the event that the candidate is a woman applying to EE, F_C is the event that the candidate is a woman applying to CS. Similarly, we can define M_E and M_C. Assumption: Candidates are either men or women, and that no candidate is allowed to be part of both EE and CS.

Lawsuit claim: Male candidate is more likely to be admitted to the university than a female i.e. $\Pr[A|M_E \cup M_C] > \Pr[A|F_E \cup F_C]$.

University response: In any given department, a male applicant is less likely to be admitted than a female i.e. $\Pr[A|F_E] > \Pr[A|M_E]$ and $\Pr[A|F_C] > \Pr[A|M_C]$.

Simpson's Paradox: Both the above statements can be simultaneously true.

Simpson's Paradox

CS	2 men admitted out of 5 candidates	40%
	50 women admitted out of 100 candidates	50%
EE	70 men admitted out of 100 candidates	70%
	4 women admitted out of 5 candidates	80%
Overall	72 men admitted, 105 candidates	$\approx$ 69%
	54 women admitted, 105 candidates	$\approx$ 51%

In the above example, $\Pr[A|F_E] = 0.8 > 0.7 = \Pr[A|M_E]$ and $\Pr[A|F_C] = 0.5 > 0.4 = \Pr[A|M_C]$. $\Pr[A|F_E \cup F_C] \approx 0.51$. Similarly, $\Pr[A|M_E \cup M_C] \approx 0.69$.

In general, Simpson's Paradox occurs when multiple small groups of data all exhibit a similar trend, but that trend reverses when those groups are aggregated.

Questions?

Back to throwing dice - Independent Events

Q: Suppose we throw two standard dice one after the other. What is the probability that we get two 6's in a row?

E = We get a 6 in the second throw. F = We get a 6 in the first throw. $E \cap F$ = we get two 6's in a row. We are computing $\Pr[E \cap F]$. $\Pr[E] = \Pr[F] = \frac{1}{6}$.

$\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]} \implies \Pr[E \cap F] = \Pr[E|F] \Pr[F]$.

Since the two dice are *independent*, knowing that we got a 6 in the first throw does not change the probability that we will get a 6 in the second throw. Hence, $\Pr[E|F] = \Pr[E]$ (conditioning does not change the probability of the event).

Hence, $\Pr[E \cap F] = \Pr[E|F] \Pr[F] = \Pr[E] \Pr[F] = \frac{1}{6} \frac{1}{6} = \frac{1}{36}$.

Independent Events

Independent Events: Events E and F are said to be independent, if knowledge that F has occurred does not change the probability that E occurs. Formally,

$$\Pr[E|F] = \Pr[E] \quad ; \quad \Pr[E \cap F] = \Pr[E] \Pr[F]$$

Q: I toss two independent, fair coins. What is the probability that I get the HT sequence?

Define E to be the event that I get a heads in the first toss, and F be the event that I get a tails in the second toss. Since the two coins are independent, events E and F are also independent. $\Pr[E \cap F] = \Pr[E] \Pr[F] = \frac{1}{2} \frac{1}{2} = \frac{1}{4}$.

Q: I randomly choose a number from $\{1, 2, \ldots, 10\}$. E is the event that the number I picked is a prime. F is the event that the number I picked is odd. Are E and F independent?

$\Pr[E] = \frac{2}{5}$, $\Pr[F] = \frac{1}{2}$, $\Pr[E \cap F] = \frac{3}{10}$. $\Pr[E \cap F] \neq \Pr[E] \Pr[F]$. Another way: $\Pr[E|F] = \frac{3}{5}$ and $\Pr[E] = \frac{2}{5}$, and hence $\Pr[E|F] \neq \Pr[E]$. Conditioning on F tell us that prime number cannot be 2, so it changes the probability of E.

Independent Events - Example

Q: We have a machine that has 2 independent components. The machine breaks if *each* of its 2 components break. Suppose each component can break with probability p, what is the probability that the machine does not break?

Let E_1 = Event that the first component breaks, E_2 = Event that the second component breaks. M = Event that the machine breaks = $E_1 \cap E_2$.

$\Pr[M] = \Pr[E_1 \cap E_2]$. Since the two components are independent, E_1 and E_2 are independent, meaning that $Pr[E_1 \cap E_2] = Pr[E_1] \Pr[E_2] = p^2$.

Probability that the machine does not break = $\Pr[M^c] = 1 - \Pr[M] = 1 - p^2$.

Independent Events - Examples

Q: We have a new machine that has 2 independent components. The machine breaks if *either* of its 2 components break. Suppose each component can break with probability p, what is the probability that the machine breaks?

For this machine, let M' be the event that it breaks. In this case, $\Pr[M'] = \Pr[E_1 \cup E_2]$.

Incorrect: By the union rule for mutually exclusive events, $\Pr[E_1 \cup E_2] = \Pr[E_1] + \Pr[E_2] = 2p$.

Mistake: *Independence does not imply mutual exclusivity* and we can not use the union rule. Independence implies that for any two events E and F, $\Pr[E \cap F] = \Pr[E] \Pr[F]$, while mutual exclusivity requires that $\Pr[E \cap F] = 0$.

Correct way:

$$\begin{aligned} \Pr[E_1 \cup E_2] &= \Pr[E_1] + \Pr[E_2] - \Pr[E_1 \cap E_2] && \text{(By the inclusion-exclusion rule)} \\ &= \Pr[E_1] + \Pr[E_2] - \Pr[E_1] \Pr[E_2] = 2p - p^2 && \text{(Since } E_1 \text{ and } E_2 \text{ are independent.)} \end{aligned}$$

Questions?

Matrix Multiplication

Given two $n \times n$ matrices – A and B, if $C = AB$, then,

$$C_{i,j} = \sum_{k=1}^{n} A_{i,k} B_{k,j}$$

Hence, in the worst case, computing $C_{i,j}$ is an $O(n)$ operation. There are n^2 entries to fill in C and hence, in the absence of additional structure, matrix multiplication takes $O(n^3)$ time.

There are non-trivial algorithms for doing matrix multiplication more efficiently:

- (Strassen, 1969) Requires $O(n^{2.81})$ operations.
- (Coppersmith-Winograd, 1987) Requires $O(n^{2.376})$ operations.
- (Alman-Williams, 2020) Requires $O(n^{2.373})$ operations.
- Belief is that it can be done in time $O(n^{2+\epsilon})$ for $\epsilon > 0$.

Verifying Matrix Multiplication

As an example, let us focus on A, B being binary 2×2 matrices.

Example: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ then $C = AB = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Objective: Verify whether a matrix multiplication operation is correct.

Trivial way: Do the matrix multiplication ourselves, and verify it using $O(n^3)$ (or $O(n^{2.373})$) operations.

Frievald's Algorithm: Randomized algorithm to verify matrix multiplication with high probability in $O(n^2)$ time.

(Basic) Frievald's Algorithm

Q: For $n \times n$ matrices A, B and D, is $D = AB$?

Algorithm:

1. Generate a random n-bit vector x, by making each bit x_i either 0 or 1 *independently* with probability $\frac{1}{2}$. E.g, for $n = 2$, toss a fair coin independently twice with the scheme – H is 0 and T is 1). If we get HT, then set $x = [0\,;\,1]$.
2. Compute $t = Bx$ and $y = At = A(Bx)$ and $z = Dx$.
3. Output "yes" if $y = z$ (all entries need to be equal), else output "no".

Computational complexity: Step 1 can be done in $O(n)$ time. Step 2 requires 3 matrix vector multiplications and can be done in $O(n^2)$ time. Step 3 requires comparing two n-dimensional vectors and can be done in $O(n)$ time. Hence, the total computational complexity is $O(n^2)$.