CMPT 210: Probability and Computing

Lecture 1

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Course Information

- **Instructor**: Sharan Vaswani (TASC-1 8221) Email: sharan_vaswani@sfu.ca
- **Office Hours**: Tuesday 11.30 am - 12.30 pm (TASC-1 8221)
- **Teaching Assistants**: Anh Dang, Matin Aghaei
- **TA Office Hours**: (From 15 Jan) Wednesday, Thursday (2.30 pm - 3.30 pm) in ASB 9814
- **Course Webpage**: https://vaswanis.github.io/210-W24.html
- **Piazza**: https://piazza.com/sfu.ca/spring2024/cmpt210/home
- **Prerequisites**: MACM 101, MATH 152 and MATH 232/MATH 240

Course Information

Objective: Introduce the foundational concepts in probability required by computing.

Syllabus:

- Combinatorics: Permutations, Binomial coefficients, Inclusion-Exclusion
- Probability theory: Independence, Conditional probability, Bayes' Theorem
- Probability theory: Random variables, Expectation, Variance
- Discrete distributions: Bernoulli, Binomial and Geometric, Joint distributions
- Tail inequalities: Markov's Inequality, Chebyshev's Inequality, Chernoff Bound
- Applications: Verifying matrix multiplication, Max-Cut, Machine Learning, Randomized QuickSort, AB Testing

Primary Resources:

- Mathematics for Computer Science (Meyer, Lehman, Leighton): `https://people.csail.mit.edu/meyer/mcs.pdf`
- Introduction to Probability and Statistics for Engineers and Scientists (Ross).

Course Information

Grading:

- 4 Assignments (45%)
- 1 Mid-Term (20%) (29 February)
- 1 Final Exam (35%) (TBD)

- Each assignment is due in 1 week via Coursys (on Tuesdays/Thursdays).
- For some flexibility, each student is allowed 1 late-submission and can submit the assignment the following Tuesday/Thursday.
- If you miss the mid-term (for a well-justified reason), we will reassign weight to the final.
- If you miss the final, there will be a make-up exam.

Questions?

Sets

Informal definition: Unordered collection of objects (referred to as *elements*)

Examples: $\{a, b, c\}$, $\{\{a, b\}, \{c, a\}\}$, $\{1.2, 2.5\}$, {yellow, red, green}, $\{x | x$ is capital of a North American country$\}$, $\{x | x$ is an integer in $[5, 10]\}$.

There is no notion of an element appearing twice. E.g. $\{a, a, b\} = \{a, b\}$.

The order of the elements does not matter. E.g. $A = \{a, b\} = \{b, a\}$.

C $= \{x | x$ is a color of the rainbow $\}$

Elements of C: red, orange, yellow, green, blue, indigo, violet.

Membership: red $\in C$, brown $\notin C$.

Cardinality: Number of elements in the set. $|C| = 7$

Q: A $= \{x | 5 < x < 17$ and x is a power of 2 $\}$. Enumerate A. What is $|A|$?

Ans: $A = \{8, 16\}$, $|A| = 2$

Common Sets

- $\emptyset$: Empty Set
- $\mathbb{N}$: Set of nonnegative integers $\{0, 1, 2 \ldots\}$
- $\mathbb{Z}$: Set of integers $\{-2, -1, 0, 1, 2 \ldots\}$
- $\mathbb{Q}$: Set of rational numbers that can be expressed as p/q where $p, q \in \mathbb{Z}$ and $q \neq 0$. $\{-10.1, -1.2, 0, 5.5, 15 \ldots\}$
- $\mathbb{R}$: Set of real numbers $\{e, \pi, \sqrt{2}, 2, 5.4\}$
- $\mathbb{C}$: Set of complex numbers $\{2 + 5i, -i, 1, 23.3, \sqrt{2}\}$

Comparing sets: A is a subset of B ($A \subseteq B$) iff every element of A is an element of B. E.g. $A = \{a, b\}$ and $B = \{a, b, c\}$, then $A \subseteq B$. Every set is a subset of itself i.e. $A \subseteq A$.

A is a *proper* subset of B ($A \subset B$) iff A is a subset of B, and A is not equal to B,

Q: Is $\{1, 4, 2\} \subset \{2, 4, 1\}$. Is $\{1, 4, 2\} \subseteq \{2, 4, 1\}$ Ans: No, Yes
Q: Is $\mathbb{N} \subset \mathbb{Z}$? Is $\mathbb{C} \subset \mathbb{R}$? Ans: Yes, No
Q: What is $|\emptyset|$? Ans: 0

Set Operations

Union: The union of sets A and B consists of elements appearing in A OR B. If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$.

Intersection: The intersection of sets A and B consists of elements that appear in both A AND B. If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cap B = \{3\}$.

Set Operations

Set difference: The set difference of A and B consists of all elements that are in A, but not in B. $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \backslash B = A - B = \{1, 2\}$. $B \backslash A = B - A = \{4, 5\}$.

Complement: Given a domain (or universe) D such that $A \subset D$, the complement of A consists of all elements that are not in A. $D = \mathbb{N}$, $A = \{1, 2, 3\}$. $A \subset D$ and $\bar{A} = \{0, 4, 5, 6, \ldots\}$.

$A \cup \bar{A} = D$, $A \cap \bar{A} = \emptyset$, $A \backslash \bar{A} = A$.

Q: $D = \mathbb{N}$, $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Compute $\overline{A \cap B}$, $(B \backslash A) \cup (A \backslash B)$.

Ans: $\overline{A \cap B} = \{0, 1, 2, 4, 5, \ldots\}$, $(B \backslash A) \cup (A \backslash B) = \{1, 2, 4, 5\}$

Power set of A is the set of all subsets of A. If $A = \{a, b, c\}$, then $\text{Pow}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Set operations and relations

Disjoint sets: Two sets are *disjoint* iff $A \cap B = \emptyset$.

Symmetric Difference: $A \Delta B$ is the set that contains those elements that are either in A or in B, but not in both.

Q: Show $A \Delta B$ on a Venn diagram. For $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, compute $A \Delta B$.

Ans: $A \Delta B = \{1, 2, 4, 5\}$

Cartesian product of sets is a set consisting of ordered pairs (*tuples*), i.e.
$A \times B = \{(a, b) \text{ s.t. } a \in A, b \in B\}$. If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$.
$A \times B = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5)\}$.

If sets are 1-dimensional objects, Cartesian product of 2 sets can be thought of as 2-dimensional.

Q. Is $A \times B = B \times A$? Ans: No. The order matters

In general, $A_1 \times A_2 \times \ldots \times A_k = \{(a_1, a_2, \ldots, a_k) | a_1 \in A_1, a_2 \in A_2, \ldots, a_k \in A_k\}$ where $(a_1, a_2, \ldots, a_k)$ is referred to as a k-tuple.

Laws of Set Theory

Distributive Law: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$z \in A \cap (B \cup C)$
iff $z \in A$ AND $z \in (B \cup C)$
iff $z \in A$ AND ($z \in B$ OR $z \in C$)
Use the distributivity of AND over OR, for binary literals $w, x, y \in \{0, 1\}$, x AND (y OR w) = (x AND y) OR (x AND w). For $x := z \in A$, $y := z \in B$, $w := z \in C$,

iff ($z \in A$ AND $z \in B$) OR ($z \in A$ AND $z \in C$)
iff $z \in (A \cap B)$ OR $z \in (A \cap C)$
iff $z \in (A \cap B) \cup (A \cap C)$

Questions?

Functions

A function assigns an element of one set, called the *domain*, to an element of another set, called the *codomain* s.t. for every element in the domain, there is at most one element in the codomain.

If A is the domain and B is the codomain of function f, then $f : A \to B$.

If $a \in A$, and $b \in B$, and $f(a) = b$, we say the function f maps a to b, b is the value of f at argument a, b is the image of a, a is the preimage of b.

$A = \{a, b, c, \ldots z\}$, $B = \{1, 2, 3, \ldots 26\}$, then we can define a function $f : A \to B$ such that $f(a) = 1$, $f(b) = 2$. f thus assigns a number to each letter in the alphabet.

Consider $f : \mathbb{R} \to \mathbb{R}$ s.t. for $x \in \mathbb{R}$, $f(x) = x^2$. $f(2.5) = 6.25 \in \mathbb{R}$.

A function cannot assign different elements in the codomain to the same element in the domain. For example, if $f(a) = 1$ and $f(a) = 2$, the f is not a function.

Functions

A function that assigns a value to every element in the domain is called a *total* function, while one that does not necessarily do so is called a *partial* function.

For $x \in \mathbb{R}$, $f(x) = 1/x^2$ is a partial function because no value is assigned to $x = 0$, since $1/0$ is undefined.

Q: Consider $f : \mathbb{R}_+ \to \mathbb{R}$ such that $f(x) = x$. Is f a function? Ans: Yes

Q: For $x \in [-1, 1], y \in \mathbb{R}$, consider $g(x) = y$ s.t. $x^2 + y^2 = 1$. Is g a function? Ans: No

Q: For $x \in \{-1, 1\}, y \in \mathbb{R}$, consider $g(x) = y$ s.t. $x^2 + y^2 = 1$. Is g a function? Ans: Yes