# CMPT 210: Probability and Computing

Lecture 8

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# Recap - Conditional Probability

For events $E$ and $F$, we wish to compute $\Pr[E|F]$, the probability of event $E$ conditioned on $F$.

**Approach 1**: With conditioning, $F$ can be interpreted as the *new sample space* such that for $\omega \notin F$, $\Pr[\omega|F] = 0$.

Example: For computing Pr(we get a 6|the outcome is even), the new sample space is $F = \{2, 4, 6\}$ and the resulting probability space is uniform. $\Pr[\{\text{even number}\}] = \frac{1}{3}$ and $\Pr[\{\text{odd number}\}] = 0$.

**Approach 2**: $\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$.

Example: $E \cap F = \{6\}$. $\Pr[E \cap F] = \frac{1}{6}$. $\Pr[F] = \Pr[\{2\}] + \Pr[\{4\}] + \Pr[\{6\}] = \frac{1}{2}$. Hence, $\frac{\Pr[E \cap F]}{\Pr[F]} = \frac{1/6}{1/2} = \frac{1}{3}$.

# Conditional Probability - Generalization to multiple events

**Multiplication Rule**: For events $E_1$, $E_2$, $E_3$, $\Pr[E_1 \cap E_2 \cap E_3] = \Pr[E_1] \Pr[E_2|E_1] \Pr[E_3|E_1 \cap E_2]$.

*Proof*:

$$\Pr[E_1] \Pr[E_2|E_1] \Pr[E_3|E_1 \cap E_2] = \Pr[E_1] \frac{\Pr[E_2 \cap E_1]}{\Pr[E_1]} \frac{\Pr[E_1 \cap E_2 \cap E_3]}{\Pr[E_1 \cap E_2]} = \Pr[E_1 \cap E_2 \cap E_3]$$

We can order the events to compute $\Pr[E_1 \cap E_2 \cap E_3]$ more easily. For example,

$$\Pr[E_1 \cap E_2 \cap E_3] = \Pr[E_2] \Pr[E_3|E_2] \Pr[E_1|E_2 \cap E_3]$$

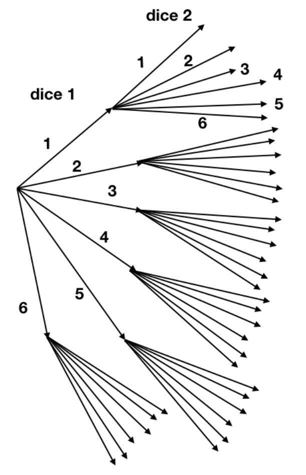
Can extend this to $n$ events i.e. in general,

$$\Pr[E_1 \cap E_2 \ldots \cap E_n] = \Pr[E_1] \Pr[E_2|E_1] \Pr[E_3|E_1 \cap E_2] \ldots \Pr[E_n|E_1 \cap E_2 \cap \ldots E_{n-1}]$$

**Questions?**

# Back to throwing dice - Tree Diagram

**Q**: Suppose we throw two standard dice one after the other. What is the probability that we get two 6's in a row?




**Identify Outcomes**: Each leaf is an outcome and $\mathcal{S} = \{(1,1),(1,2),(1,3),\ldots(6,6)\}$.

**Identify Event**: $E = \{(6,6)\}$.

**Compute probabilities**: $\Pr[\text{Dice 1 is 6}] = \frac{1}{6}$.
$\Pr[(6,3)] = \Pr[\text{Dice 2 is 3} \ \cap \ \text{Dice 1 is 6}] = \Pr[\text{Dice 2 is 3} \mid \text{Dice 1 is 6}] \Pr[\text{Dice 1 is 6}] = \frac{1}{6}\frac{1}{6} = \frac{1}{36}$.

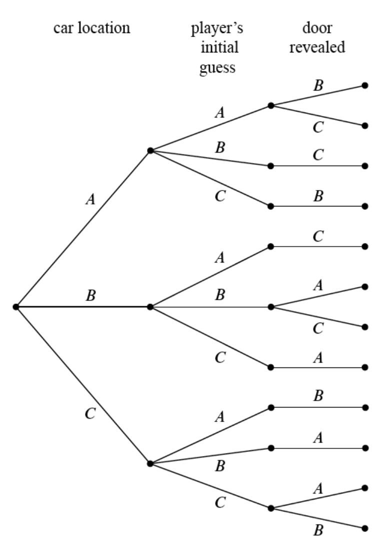
$\Pr[E] = \Pr[\text{dice 1 is 6} \ \cap \text{dice 2 is 6} \ ] = \frac{1}{36}$.

# Monty Hall Problem

**Q**: Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say A, and the host, who knows what's behind the doors, opens another door, say C, which has a goat. He says to you, "Do you want to pick door B?" Is it to your advantage to switch your choice of doors?

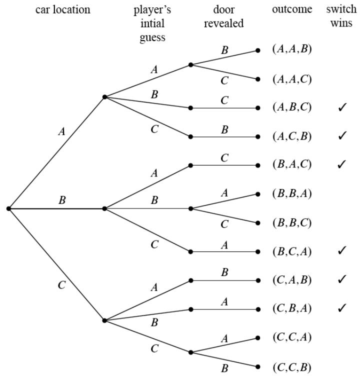
- The car is equally likely to be hidden behind each of the three doors.
- The player is equally likely to pick each of the three doors, regardless of the car's location.
- After the player picks a door, the host must open a different door with a goat behind it and offer the player the choice of staying with the original door or switching.
- If the host has a choice of which door to open, then he is equally likely to select each of them.

# Tree Diagram for the Monty Hall Problem - Identify Outcomes




$\mathcal{S} = \{(A, A, B), (A, A, C), (A, B, C), (A, C, B), \ldots\}.$
$E_1 =$ Prize is behind door C $=$
$\{(C, A, B), (C, B, A), (C, C, A), (C, C, B)\}$

# Tree Diagram for the Monty Hall Problem - Identify Event

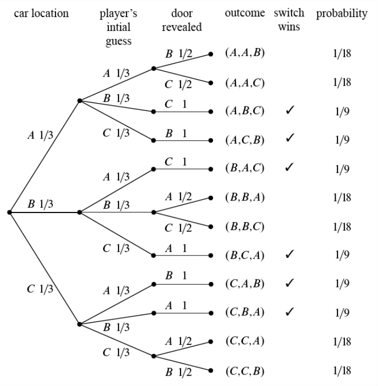



$E$ = Switching wins =
$\{(A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B), (C, B, A)\}$

$\Pr[(A, A)] = \Pr[\text{Car is at A} \cap \text{Player picks A}] =$
$\Pr[\text{Player picks A} \mid \text{Car is at A}] \Pr[\text{Car is at A}] = \frac{1}{3}\frac{1}{3} = \frac{1}{9}.$

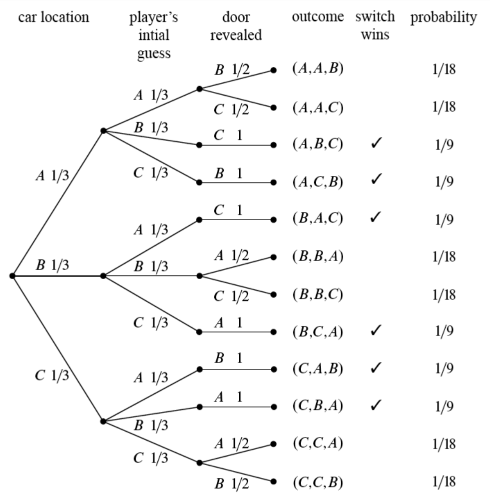
$\Pr[(A, A, B)] = \Pr[\text{Door B is revealed} \cap \text{AA}] =$
$\Pr[\text{Door B is revealed} \mid \text{AA}] \Pr[\text{AA}] = \frac{1}{2}\frac{1}{9} = \frac{1}{18}.$

# Tree Diagram for the Monty Hall Problem - Compute Probabilities




$$\Pr[E] = \Pr[(A,B,C)] + \Pr[(A,C,B)] + \Pr[(B,A,C)] + \Pr[(B,C,A)] + \Pr[(C,A,B)] + \Pr[(C,B,A)] = \frac{1}{9} \times 6 = \frac{2}{3}.$$

# Monty Hall Problem and Conditional Probability




**Q**: What is the probability of winning by switching, if we pick door A and door B is opened.

$$\Pr[\text{win by switching}|\text{pick A and door B is opened}] = \frac{\Pr[\text{win by switching} \cap \text{pick A and door B is opened}]}{\Pr[\text{pick A and door B is opened}]}$$

$$= \frac{\Pr[(C,A,B)]}{\Pr[\{(A,A,B),(C,A,B)\}]} = \frac{1/9}{1/9+1/18} = \frac{2}{3}.$$

Q: Compute

$\Pr[\text{win by switching}|\text{pick A and door C is opened}]$?

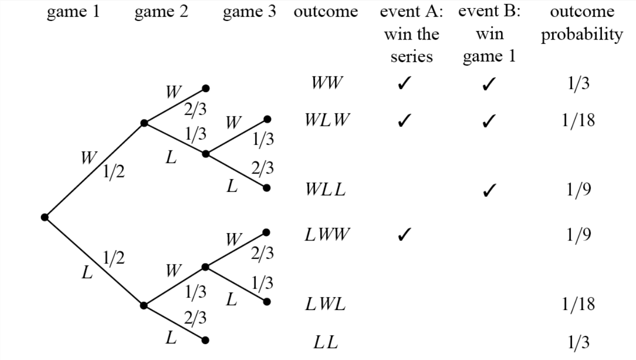
Ans: $\frac{\Pr[\text{win by switching} \cap \text{pick A and door C is opened}]}{\Pr[\text{pick A and door C is opened}]}$

$$= \frac{\Pr[(B,A,C)]}{\Pr[\{(A,A,C),(B,A,C)\}]} = \frac{1/9}{1/9+1/18} = \frac{2}{3}.$$

Questions?

**Q**: In a best-of-three series, the local hockey team wins the first game with probability $\frac{1}{2}$. In subsequent games, their probability of winning is determined by the outcome of the previous game. If the team won the previous game, then they are invigorated by victory and win the current game with probability $\frac{2}{3}$. If they lost the previous game, then they are demoralized by defeat and win the current game with probability only $\frac{1}{3}$. What is the probability that the local team wins the series, given that they win the first game? Note that the series is over as soon as a team wins two games.

# Conditional Probability - Examples

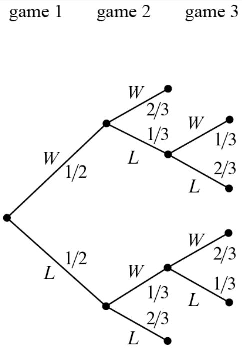



**Sample space**: $\mathcal{S} = \{(W,W),(W,L,W),(W,L,L),(L,W,W),(L,W,L),(L,L)\}$.

**Events**: $T = \{(W,W),(W,L,W),(L,W,W)\}$, $F = \{(W,W),(W,L,W),(W,L,L)\}$.

$$\Pr[T|F] = \frac{\Pr[T \cap F]}{\Pr[F]} = \frac{\Pr[\{(W,W),(W,L,W)\}]}{\Pr[\{(W,W),(W,L,W),(W,L,L)\}]} = \frac{1/3+1/18}{1/3+1/18+1/9} = \frac{7}{9}$$

# Conditional Probability - Examples

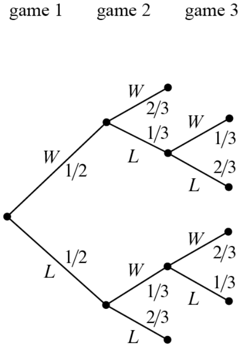



| outcome | event A: win the series | event B: win game 1 | outcome probability |
|---|---|---|---|
| $WW$ | ✓ | ✓ | 1/3 |
| $WLW$ | ✓ | ✓ | 1/18 |
| $WLL$ | | ✓ | 1/9 |
| $LWW$ | ✓ | | 1/9 |
| $LWL$ | | | 1/18 |
| $LL$ | | | 1/3 |

Q: What is the probability that the team wins the series if they lose Game 1? Ans: $\frac{1/9}{1/9+1/18+1/3} = \frac{2}{9}$

Q: What is the probability that the team wins the series? Ans: $\frac{1}{2}$

Q: What is the probability that the series goes to Game 3? Ans: $\frac{1}{3}$



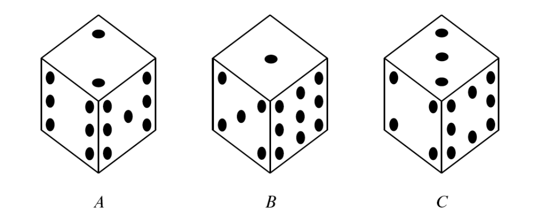

| outcome | event A: win the series | event B: win game 1 | outcome probability |
|---|---|---|---|
| $WW$ | ✓ | ✓ | 1/3 |
| $WLW$ | ✓ | ✓ | 1/18 |
| $WLL$ | | ✓ | 1/9 |
| $LWW$ | ✓ | | 1/9 |
| $LWL$ | | | 1/18 |
| $LL$ | | | 1/3 |

**Q**: What is the probability that the team won their first game given that they won the series?

Recall that $T = \{(W, W), (W, L, W), (L, W, W)\}$, $F = \{(W, W), (W, L, W), (W, L, L)\}$. We wish to compute $\Pr[F|T] = \frac{\Pr[F \cap T]}{\Pr[T]} = \frac{\Pr[\{(W,W),(W,L,W)\}]}{\Pr[\{(W,W),(W,L,W),(L,W,W)\}]} = \frac{1/3+1/18}{1/3+1/18+1/9} = \frac{7}{9}$.

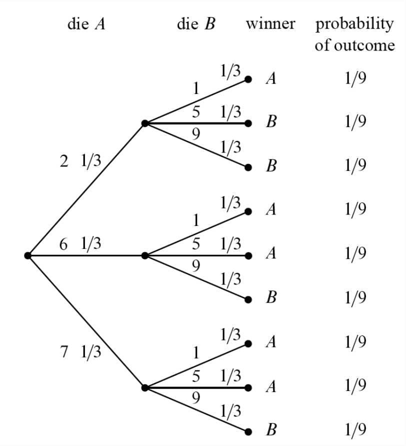
# Conditional Probability - Examples

Let us play a game with three strange dice shown in the figure. Each player selects one die and rolls it once. The player with the lower value pays the other player \$100. We can pick a die first, after which the other player can pick one of the other two.




**Q**: Suppose we choose die B because it has a 9, and the other player selects die A. What is the probability that we will win?

# Conditional Probability - Examples




**Identify Outcomes**: Each leaf is an outcome and $\mathcal{S} = \{(2,1),(2,5),(2,9),(6,1),(6,5),(6,9),(7,1),(7,5),(7,9)\}$.

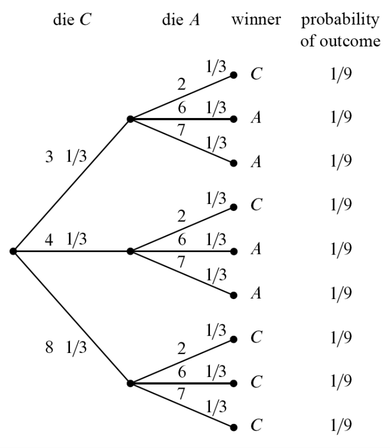
**Identify Event**: $E = \{(2,5),(2,9),(6,9),(7,9)\}$.

**Compute probabilities**: $\Pr[\text{Dice 1 is 6}] = \frac{1}{3}$.
$\Pr[(6,5)] = \Pr[\text{Dice 2 is 5} \,\cap\, \text{Dice 1 is 6}] = \Pr[\text{Dice 2 is 5} \mid \text{Dice 1 is 6}]\,\Pr[\text{Dice 1 is 6}] = \frac{1}{3}\frac{1}{3} = \frac{1}{9}$.

$\Pr[E] = \Pr[(2,5)] + \Pr[(2,9)] + \Pr[(6,9)] + \Pr[(7,9)] = \frac{4}{9}$.
Meaning that there is less than 50% chance of winning.

# Conditional Probability - Examples

**Q**: We get another chance – this time we know that die A is good (since we lost to it previously), we choose die A and the other player chooses die C. What is our probability of winning?




Now, $E = \{(3,6),(3,7),(4,6),(4,7)\}$ and hence $\Pr[E] = \frac{4}{9}$. Meaning that there is less than 50% chance of winning.

# Conditional Probability - Examples

We get yet another chance, and this time we choose die C, because we reason that die A is better than B, and C is better than A.

We can construct a similar tree diagram to show that the probability that we win is again $\frac{4}{9}$.

- A beats B with probability $\frac{5}{9}$ (first game).
- C beats A with probability $\frac{5}{9}$ (second game).
- B beats C with probability $\frac{5}{9}$ (third game).

Since A will beat B more often than not, and B will beat C more often than not, it seems like A ought to beat C more often than not, that is, the "beats more often" relation ought to be transitive. But this intuitive idea is false: whatever die we pick, the second player can pick one of the others and be likely to win. So picking first is actually a disadvantage!

This is the topic of some recent research and was covered in this article:
`https://www.quantamagazine.org/mathematicians-roll-dice-and-get-rock-paper-scissors-20230119/`