CMPT 210: Probability and Computing

Lecture 7

Sharan Vaswani

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Recap - Conditional Probability

For events E and F, we wish to compute $\Pr[E|F]$, the probability of event E conditioned on F.

With conditioning, F can be interpreted as the *new sample space* such that for $\omega \notin F$, $\Pr[\omega|F] = 0$.

Example: For computing $\Pr[\text{we get a } 6|\text{the outcome is even}]$, the new sample space is $F = \{2, 4, 6\}$ and the resulting probability space is uniform. $\Pr[\{\text{even number}\}] = \frac{1}{3}$ and $\Pr[\{\text{odd number}\}] = 0$.

Conditional Probability

Conditional Probability Rule: For two events E and F, $\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$, where $\Pr[F] \neq 0$.

Proof: By conditioning on F, the only outcomes we care about are in F i.e. for $\omega \notin F$, $\Pr[\omega|F] = 0$.

Since we want to compute the probability that event E happens, we care about the outcomes that are in E. Hence, the outcomes we care about lie in both E *and* F, meaning that $\omega \in E \cap F$.
$\implies \Pr[E|F] \propto \sum_{\omega \in (E \cap F)} \Pr[\omega]$. By definition of proportionality, for some constant $c > 0$, $\Pr[E|F] = c \sum_{\omega \in (E \cap F)} \Pr[\omega]$.

We know that $\Pr[F|F] = 1$ (probability of event F given that F has happened). Hence, $\Pr[F|F] = 1 = c \sum_{\omega \in F} \Pr[\omega] \implies c = \frac{1}{\sum_{\omega \in F} \Pr[\omega]}$.

Substituting the value of c,

$$\Pr[E|F] = \frac{\sum_{\omega \in (E \cap F)} \Pr[\omega]}{\sum_{\omega \in F} \Pr[\omega]} = \frac{\Pr[E \cap F]}{\Pr[F]}, \text{where } \Pr[F] \neq 0.$$

This formula gives an alternate way to compute conditional probabilities.

Back to throwing dice

Q: Suppose we throw a "standard" dice, what is the probability of getting a 6 if we are told that the outcome is even?

Sample space: $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ and $\Pr[1] = \Pr[2] = \ldots = \Pr[6] = \frac{1}{6}$.
Event: $E = \{6\}$. We are conditioning on $F = \{2, 4, 6\}$.
$\Pr[\text{we get a } 6 | \text{the outcome is even}] = \Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$.
$E \cap F = \{6\}$. $\Pr[E \cap F] = \frac{1}{6}$. $\Pr[F] = \Pr[\{2\}] + \Pr[\{4\}] + \Pr[\{6\}] = \frac{1}{2}$.
Hence, $\frac{\Pr[E \cap F]}{\Pr[F]} = \frac{1/6}{1/2} = \frac{1}{3}$.

Q: What is the probability of getting either a 3 or 6 if we are told that the outcome is even? Ans: $E = \{3, 6\}$, $F = \{2, 4, 6\}$. $E \cap F = \{6\}$. $\Pr[E] = \Pr[\{6\}] = \frac{1}{6}$. $\Pr[\{F\}] = \frac{1}{2}$. Hence, $\Pr[E|F] = \frac{1}{3}$.

Q: Suppose we throw two standard dice one after the other. What is the probability that the sum of the dice is 6 given that the first dice came up 4? Ans: $S = \{(1, 1), (1, 2), (1, 3), \ldots, (6, 6)\}$, $E = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$, $F = \{(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)\}$.
$\Pr[F] = \frac{6}{36}$, $E \cap F = \{(4, 2)\}$ $\Pr[E \cap F] = \frac{1}{36}$, $\Pr[E|F] = \frac{1/36}{6/36} = \frac{1}{6}$

Questions?

Conditional Probability - Examples

Q: Suppose we select a card at random from a standard deck of 52 cards. What is the probability of getting:

- A spade conditioned on the event that I picked the red color Ans: 0
- A spade facecard conditioned on the event that I picked the black color Ans: $\frac{3}{26}$
- A black card conditioned on the event that I picked a spade facecard Ans: 1
- The queen of hearts given that I picked a queen Ans: $\frac{1}{4}$
- An ace given that I picked a spade Ans: $\frac{1}{13}$

Conditional Probability - Examples

Q: The organization that Jones works for is running a father–son dinner for those employees having at least one son. Each of these employees is invited to attend along with his youngest son. If Jones is known to have two children, what is the conditional probability that they are both boys given that he is invited to the dinner? Assume that the sample space S is given by $S = \{(b, b), (b, g), (g, b), (g, g)\}$ and all outcomes are equally likely. For instance, (b, g) means that the younger child is a boy and the older child is a girl.

The event that we care about is Jones has both boys. Hence, $E = \{(b, b)\}$.

Additional information that we are conditioning on is that Jones is invited to the dinner meaning that he has at least one son. Hence, $F = \{(b, b), (b, g), (g, b)\}$.

Hence, $E \cap F = \{(b, b)\}$, $\Pr[E \cap F] = \frac{|E \cap F|}{|\mathcal{S}|} = \frac{1}{4}$. $\Pr[F] = \frac{|F|}{|\mathcal{S}|} = \frac{3}{4}$.

$\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]} = \frac{1/4}{3/4} = \frac{1}{3}$.

Conditional Probability - Examples

Q: Ms. Perez figures that there is a 30 percent chance that her company will set up a branch office in Phoenix. If it does, she is 60 percent certain that she will be made manager of this new operation. What is the probability that there will be a branch in Phoenix and Perez will be its office manager?

E = Perez will be a branch office manager; F = her company will set up a branch office in Phoenix; $E \cap F$ = Perez will be an office manager in the Phoenix branch.

From the question, we know that $\Pr[F] = 0.3$, $\Pr[E|F] = 0.6$. Hence,
$\Pr[E \cap F] = \Pr[E] \Pr[E|F] = 0.3 \times 0.6 = 0.18$.

Conditional Probability Examples

Q: Suppose we have a bowl containing 6 white and 5 black balls. We randomly draw a ball. What is the probability that we draw a black ball? Ans: $\frac{5}{11}$

Q: We randomly draw two balls, one after the other (without putting the first back). What is the probability that we (i) draw a black ball followed by a white ball (ii) draw a white ball followed by a black ball (iii) we get one black ball and one white ball (iv) both black (v) both white?

B1 = Draw black first, W1 = Draw white first. B2 = Black second, W2 = White second.

(i) $\Pr[B1] = \frac{5}{11}$. $\Pr[W2|B1] = \frac{6}{10}$. Hence, $\Pr[B1 \cap W2] = \Pr[B1]\Pr[W2|B1] = \frac{30}{110}$.

(ii) $\Pr[W1] = \frac{6}{11}$. $\Pr[B2|W1] = \frac{5}{10}$. Hence, $\Pr[W1 \cap B2] = \Pr[W1]\Pr[B2|W1] = \frac{30}{110}$.

(iii) $G = (B1 \cap W2) \cup (W1 \cap B2)$. Events $B1 \cap W2$ and $B2 \cap W1$ are mutually exclusive. By the union rule for mutually exclusive events, $\Pr[G] = \Pr[B1 \cap W2] + \Pr[W1 \cap B2] = \frac{60}{110}$.

(iv) $\Pr[B1 \cap B2] = \Pr[B1]\Pr[B2|B1] = \frac{20}{110}$.

(v) $\Pr[W1 \cap W2] = \Pr[W1]\Pr[W2|W1] = \frac{30}{110}$.