CMPT 210: Probability and Computing

Lecture 15

Sharan Vaswani

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Recap

Expectation/mean of a random variable R is denoted by $\mathbb{E}[R]$ and "summarizes" its distribution. Formally, $\mathbb{E}[R] := \sum_{\omega \in \mathcal{S}} \Pr[\omega] \, R[\omega]$

Example: When throwing a standard dice, if R is the random variable equal to the number on the dice. $\mathbb{E}[R] = \sum_{i \in \{1,2,\ldots,6\}} \frac{1}{6}[i] = \frac{7}{2}$.

Alternate definition of expectation: $\mathbb{E}[R] = \sum_{x \in \text{Range}(R)} x \, \Pr[R = x]$.

This definition does not depend on the sample space.

Example: If $\mathcal{I}_A$ is the indicator random variable for event A, then $\mathbb{E}[\mathcal{I}_A] = \Pr[\mathcal{I}_A = 1](1) + \Pr[\mathcal{I}_A = 0](0) = \Pr[A]$. For $\mathcal{I}_A$, the expectation is equal to the probability that event A happens.

Linearity of Expectation: For n random variables $R_1, R_2, \ldots, R_n$ and constants $a_1, a_2, \ldots, a_n$, $\mathbb{E}\left[\sum_{i=1}^{n} a_i R_i\right] = \sum_{i=1}^{n} a_i \, \mathbb{E}[R_i]$.

Recap

If $R \sim \text{Bernoulli}(p)$, $\mathbb{E}[R] = p$. *Example*: When tossing a coin, if R is the random variable equal to 1 if we get a heads.

If $R \sim \text{Uniform}(\{v_1, \ldots, v_n\})$, $\mathbb{E}[R] = \frac{v_1 + v_2 + \ldots + v_n}{n}$. *Example*: When throwing an n-sided dice with numbers $v_1, \ldots v_n$, if R is the random variable equal to the number.

If $R \sim \text{Bin}(n, p)$, $\mathbb{E}[R] = np$. *Example*: When tossing n independent coins, if R is the random variable equal to the number of heads.

If $R \sim \text{Geo}(p)$, $\mathbb{E}[R] = \frac{1}{p}$. *Example*: When tossing a coin repeatedly, if R is the random variable equal to the number of tosses required to get the first heads.

Expectation - Examples - Coupon Collector Problem

Q: In a game started by a coffee shop, each time we buy a coffee, we get a coupon. Each coupon has a color (amongst n different colors) and each time, the color of the coupon is selected uniformly at random from amongst the n colors. If we collect at least one coupon of each color, we can claim a free coffee. On average, how many coupons should we collect (coffees we should buy) to claim the prize?

Suppose we get the following sequence of coupons:

$$blue, green, green, red, blue, orange, blue, orange, gray$$

Let us partition this sequence into segments such that a segment ends when we collect a coupon of a new color we did not have before. For this example,

$$\underbrace{blue}_{S_1}\ \underbrace{green}_{S_2}\ \underbrace{green, red}_{S_3}\ \underbrace{blue, orange}_{S_4}\ \underbrace{blue, orange, gray}_{S_5}$$

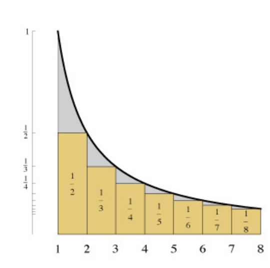
If the number of segments is equal to n, by definition, we will have collected coupons of the n different colors. Define X_k to be the random variable equal to the length of segment S_k and T to be the total number of coupons required to have at least one coupon per color.

Expectation - Examples - Coupon Collector Problem

$T = X_1 + X_2 + \ldots X_n$. We wish to compute $\mathbb{E}[T]$. By linearity of expectation, $\mathbb{E}[T] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \ldots + \mathbb{E}[X_n]$.

Let us calculate $\mathbb{E}[X_k]$. If we are on segment k, we have seen $k-1$ colors before. Hence, the probability of seeing a new (one that we have not seen before) colored coupon in S_k is $\frac{n-(k-1)}{n}$. $X_k \sim \text{Geo}\left(\frac{n-(k-1)}{n}\right)$, and we know that $\mathbb{E}[X_k] = \frac{n}{n-k+1}$.

$$\mathbb{E}[T] = \sum_{k=1}^{n} \frac{n}{n-k+1} = n\left[\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{1}\right]$$
$$\leq n\left[1 + \int_1^n \frac{dx}{x}\right] = n\left[1 + \ln(n)\right]$$

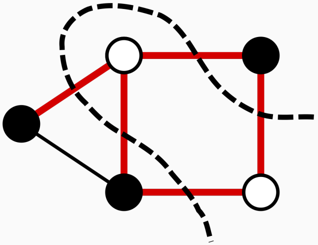


We also know that $\mathbb{E}[T] \geq n\ln(n+1)$. Hence, $\mathbb{E}[T] = O(n\ln(n))$, meaning that we need to buy $O(n\ln(n))$ coffees to collect coupons of n colors and get a free coffee.

Questions?

Max Cut

Given a graph $G = (\mathcal{V}, \mathcal{E})$, partition the graph's vertices into two complementary sets $\mathcal{S}$ and $\mathcal{T}$, such that the number of edges between the set $\mathcal{S}$ and the set $\mathcal{T}$ is as large as possible.



Max Cut has applications to VLSI circuit design.

Equivalently, find a set $\mathcal{U} \subseteq \mathcal{V}$ of vertices that solve the following

$$\max_{\mathcal{U} \subseteq \mathcal{V}} |\delta(\mathcal{U})| \text{ where } \delta(\mathcal{U}) := \{(u, v) \in \mathcal{E} | u \in \mathcal{U} \text{ and } v \notin \mathcal{U}\}$$

Here, $\delta(\mathcal{U})$ is referred to as the "cut" corresponding to the set $\mathcal{U}$.

Max Cut

- Max Cut is NP-hard (Karp, 1972), meaning that there is no polynomial (in $|\mathcal{E}|$) time algorithm that solves Max Cut exactly.
- We want to find an approximate solution $\mathcal{U}$ such that, if OPT is the size of the optimal cut, then, $|\delta(\mathcal{U})| \geq \alpha\, OPT$ where $\alpha \in (0, 1)$ is the multiplicative approximation factor.
- Randomized algorithm that guarantees an approximate solution with $\alpha = \frac{1}{2}$ with probability close to 1 (Erdos, 1967).
- Algorithm with $\alpha = 0.878$. (Goemans and Williamson, 1995).
- Under some technical conditions, no efficient algorithm has $\alpha > 0.878$ (Khot et al, 2004).

We will use Erdos' randomized algorithm and first prove the result in expectation. We wish to prove that for $\mathcal{U}$ returned by Erdos' algorithm,

$$\mathbb{E}[|\delta(\mathcal{U})|] \geq \frac{1}{2} OPT$$

Algorithm: Select $\mathcal{U}$ to be a random subset of $\mathcal{V}$ i.e. for each vertex v, choose v to be in the set $\mathcal{U}$ independently with probability $\frac{1}{2}$ (do not even look at the edges!).

Max Cut

Claim: For Erdos' algorithm, $\mathbb{E}[|\delta(\mathcal{U})|] \geq \frac{1}{2} OPT$.

Proof: For each edge $(u, v) \in \mathcal{E}$, let $X_{u,v}$ be the indicator random variable equal to 1 iff the event $E_{u,v} = \{(u, v) \in \delta(\mathcal{U})\}$ happens.

$$\mathbb{E}[|\delta(\mathcal{U})|] = \mathbb{E}\left[\sum_{(u,v)\in\mathcal{E}} X_{u,v}\right] = \sum_{(u,v)\in\mathcal{E}} \mathbb{E}\left[X_{u,v}\right] = \sum_{(u,v)\in\mathcal{E}} \Pr[E_{u,v}]$$

(Linearity of expectation, and Expectation of indicator r.v's.)

$$\Pr[E_{u,v}] = \Pr[(u, v) \in \delta(\mathcal{U})] = \Pr\left[(u \in \mathcal{U} \cap v \notin \mathcal{U}) \cup (u \notin \mathcal{U} \cap v \in \mathcal{U})\right]$$

$$= \Pr\left[(u \in \mathcal{U} \cap v \notin \mathcal{U})\right] + \Pr\left[(u \notin \mathcal{U} \cap v \in \mathcal{U})\right]$$ (Union rule for mutually exclusive events)

$$\Pr[E_{u,v}] = \Pr[u \in \mathcal{U}] \Pr[v \notin \mathcal{U}] + \Pr[u \notin \mathcal{U}] \Pr[v \in \mathcal{U}] = \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{2} = \frac{1}{2}.$$

(Independent events)

$$\implies \mathbb{E}[|\delta(\mathcal{U})|] = \sum_{(u,v)\in\mathcal{E}} \Pr[E_{u,v}] = \frac{|\mathcal{E}|}{2} \geq \frac{\text{OPT}}{2}.$$

Questions?

Conditional Expectation

Similar to probabilities, expectations can be conditioned on some event.

For random variable R, the expected value of R conditioned on an event A is given by:

$$\mathbb{E}[R|A] = \sum_{x \in \text{Range}(R)} x \Pr[R = x|A]$$

Q: If we throw a standard dice and define R to be the random variable equal to the number that comes up, what is the expected value of R given that the number is at most 4?

Let A be the event that the number is at most 4.
$\Pr[R = 1|A] = \frac{\Pr[(R=1) \cap A]}{\Pr[A]} = \frac{\Pr[R=1]}{\Pr[A]} = \frac{1/6}{4/6} = 1/4$.
$\Pr[R = 2|A] = \Pr[R = 3|A] = \Pr[R = 4|A] = \frac{1}{4}$ and $\Pr[R = 5|A] = \Pr[R = 6|A] = 0$.

$$\mathbb{E}[R|A] = \sum_{x \in \{1,2,3,4\}} x \Pr[R = x|A] = \frac{1}{4}[1 + 2 + 3 + 4] = \frac{5}{2}.$$

Q: What is the expected value of R given that the number is at least 4? Ans:
$\mathbb{E}[R|A] = \sum_{x \in \{4,5,6\}} x \Pr[R = x|A] = \frac{1}{3}[4 + 5 + 6] = 5$.

Law of Total Expectation

If R is a random variable $\mathcal{S} \to V$ and events $A_1, A_2, \ldots A_n$ form a partition of the sample space i.e. for all i, j, $A_i \cap A_j = \emptyset$ and $A_1 \cup A_2 \cup \ldots \cup A_n = \mathcal{S}$, then,

$$\mathbb{E}[R] = \sum_i \mathbb{E}[R|A_i] \Pr[A_i] \,.$$

Proof:

$$\mathbb{E}[R] = \sum_{x \in \text{Range}(R)} x \Pr[R = x] = \sum_{x \in \text{Range}(R)} x \sum_i \Pr[R = x|A_i] \Pr[A_i]$$

(Law of total probability)

$$= \sum_i \Pr[A_i] \sum_{x \in \text{Range}(R)} x \Pr[R = x|A_i]$$

$$\implies \mathbb{E}[R] = \sum_i \Pr[A_i] \, \mathbb{E}[R|A_i].$$

Conditional Expectation - Examples

Q: Suppose that 49.6% of the people in the world are male and the rest female. If the expected height of a randomly chosen male is 5 feet 11 inches, while the expected height of a randomly chosen female is 5 feet 5 inches, what is the expected height of a randomly chosen person?

Define H to be the random variable equal to the height (in feet) of a randomly chosen person.
Define M to be the event that the person is male and F the event that the person is female.
We wish to compute $\mathbb{E}[H]$ and we know that $\mathbb{E}[H|M] = 5 + \frac{11}{12}$ and $\mathbb{E}[H|F] = 5 + \frac{5}{12}$.
$\Pr[M] = 0.496$ and $\Pr[F] = 1 - 0.496 = 0.504$.
Hence, $\mathbb{E}[H] = \mathbb{E}[H|M] \Pr[M] + \mathbb{E}[H|F] \Pr[F] = \frac{71}{12}(0.496) + \frac{65}{12}(0.504)$.

Questions?