CMPT 210: Probability and Computing

Lecture 1

Sharan Vaswani

January 5, 2023

Course Information

- **Instructor**: Sharan Vaswani (TASC-1 8221) Email: `sharan_vaswani@sfu.ca`
- **Office Hours**: Thursday 2.30 pm - 3.30 pm (TASC-1 8221)
- **Teaching Assistant**: Anh Dang. Email: `anh_dang@sfu.ca`
- **Tutorials**: (From 16 Jan) Monday (1:30 pm - 2:20 pm, 3.30 pm - 4.20 pm) in BLU 10921
- **Course Webpage**: `https://vaswanis.github.io/210-W23`
- **Piazza**: `https://piazza.com/sfu.ca/spring2023/cmpt210/home`
- **Prerequisites**: MACM 101, MATH 152 and MATH 232/MATH 240

Course Information

Objective: Introduce the foundational concepts in probability required by computing.

Syllabus:

- Combinatorics: Permutations, Binomial coefficients, Inclusion-Exclusion
- Probability theory: Independence, Conditional probability, Bayes' Theorem
- Probability theory: Random variables, Expectation, Variance
- Discrete distributions: Bernoulli, Binomial and Geometric, Joint distributions
- Tail inequalities: Markov's Inequality, Chebyshev's Inequality, Chernoff Bound
- Applications: Verifying matrix multiplication, Max-Cut, Machine Learning, Randomized QuickSort, AB Testing
- Continuous distributions (Introduction): Normal Distribution, Central Limit Theorem

Primary Resources:

- Mathematics for Computer Science (Meyer, Lehman, Leighton): `https://people.csail.mit.edu/meyer/mcs.pdf`
- Introduction to Probability and Statistics for Engineers and Scientists (Ross).

Course Information

Grading:

- 5 Assignments ($5 \times 10\% = 50\%$)
- 1 Mid-Term ($1 \times 15\% = 15\%$) (17 February)
- 1 Final Exam ($1 \times 35\% = 35\%$) (TBD)

- Each assignment is due in 1 week (on Fridays).
- For some flexibility, each student is allowed 1 late-submission and can submit the assignment in the tutorial session (the Monday after).
- Solutions will be released on Monday evenings after the tutorial, and no late submissions are allowed after that.
- If you miss the mid-term (for a well-justified reason), will reassign weight to the final.
- If you miss the final, there will be a make-up exam.

Questions?

Sets

Informal definition: Unordered collection of objects (referred to as *elements*)

Examples: $\{a, b, c\}$, $\{\{a, b\}, \{c, a\}\}$, $\{1.2, 2.5\}$, {yellow, red, green}, $\{x | x$ is capital of a North American country$\}$, $\{x | x$ is an integer in $[5, 10]\}$.

There is no notion of an element appearing twice. E.g. $\{a, a, b\} = \{a, b\}$.

The order of the elements does not matter. E.g. $A = \{a, b\} = \{b, a\}$.

C $= \{x | x$ is a color of the rainbow $\}$

Elements of C: red, orange, yellow, green, blue, indigo, violet.

Membership: red $\in C$, brown $\notin C$.

Cardinality: Number of elements in the set. $|C| = 7$

Q: A $= \{x | 5 < x < 17$ and x is a power of 2 $\}$. Enumerate A. What is $|A|$?

Ans: $A = \{8, 16\}$, $|A| = 2$

Common Sets

- $\emptyset$: Empty Set
- $\mathbb{N}$: Set of nonnegative integers $\{0, 1, 2 \ldots\}$
- $\mathbb{Z}$: Set of integers $\{-2, -1, 0, 1, 2 \ldots\}$
- $\mathbb{Q}$: Set of rational numbers that can be expressed as $^p/_q$ where $p, q \in \mathbb{Z}$ and $q \neq 0$. $\{-10.1, -1.2, 0, 5.5, 15 \ldots\}$
- $\mathbb{R}$: Set of real numbers $\{e, \pi, \sqrt{2}, 2, 5.4\}$
- $\mathbb{C}$: Set of complex numbers $\{2 + 5i, -i, 1, 23.3, \sqrt{2}\}$

Comparing sets: A is a subset of B ($A \subseteq B$) iff every element of A is an element of B. E.g. $A = \{a, b\}$ and $B = \{a, b, c\}$, then $A \subseteq B$. Every set is a subset of itself i.e. $A \subseteq A$.

A is a *proper* subset of B ($A \subset B$) iff A is a subset of B, and A is not equal to B,

Q: Is $\{1, 4, 2\} \subset \{2, 4, 1\}$. Is $\{1, 4, 2\} \subseteq \{2, 4, 1\}$ Ans: No, Yes
Q: Is $\mathbb{N} \subset \mathbb{Z}$? Is $\mathbb{C} \subset \mathbb{R}$? Ans: Yes, No
Q: What is $|\emptyset|$? Ans: 0

Set Operations

Union: The union of sets A and B consists of elements appearing in A OR B. If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$.

Intersection: The intersection of sets A and B consists of elements that appear in both A AND B. If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cap B = \{3\}$.

Set Operations

Set difference: The set difference of A and B consists of all elements that are in A, but not in B. $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \backslash B = A - B = \{1, 2\}$. $B \backslash A = B - A = \{4, 5\}$.

Complement: Given a domain (or universe) D such that $A \subset D$, the complement of A consists of all elements that are not in A. $D = \mathbb{N}$, $A = \{1, 2, 3\}$. $A \subset D$ and $\bar{A} = \{0, 4, 5, 6, \ldots\}$.

$A \cup \bar{A} = D$, $A \cap \bar{A} = \emptyset$, $A \backslash \bar{A} = A$.

Q: $D = \mathbb{N}$, $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Compute $\overline{A \cap B}$, $(B \backslash A) \cup (A \backslash B)$.

Ans: $\overline{A \cap B} = \{0, 1, 2, 4, 5, \ldots\}$, $(B \backslash A) \cup (A \backslash B) = \{1, 2, 4, 5\}$

Power set of A is the set of all subsets of A. If $A = \{a, b, c\}$, then $\mathrm{Pow}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Set operations and relations

Disjoint sets: Two sets are *disjoint* iff $A \cap B = \emptyset$.

Symmetric Difference: $A \Delta B$ is the set that contains those elements that are either in A or in B, but not in both.

Q: Show $A \Delta B$ on a Venn diagram. For $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, compute $A \Delta B$.

Ans: $A \Delta B = \{1, 2, 4, 5\}$

Cartesian product of sets is a set consisting of ordered pairs (*tuples*), i.e.
$A \times B = \{(a, b) \text{ s.t. } a \in A, b \in B\}$. If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$.
$A \times B = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5)\}$.

If sets are 1-dimensional objects, Cartesian product of 2 sets can be thought of as 2-dimensional.

Q. Is $A \times B = B \times A$? Ans: No. The order matters

In general, $A_1 \times A_2 \times \ldots \times A_k = \{(a_1, a_2, \ldots, a_k) | a_1 \in A_1, a_2 \in A_2, \ldots, a_k \in A_k\}$ where $(a_1, a_2, \ldots, a_k)$ is referred to as a k-tuple.

Laws of Set Theory

Distributive Law: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$z \in A \cap (B \cup C)$
iff $z \in A$ AND $z \in (B \cup C)$
iff $z \in A$ AND ($z \in B$ OR $z \in C$)
Use the distributivity of AND over OR, for binary literals $w, x, y \in \{0, 1\}$, x AND (y OR w) = (x AND y) OR (x AND w). For $x := z \in A$, $y := z \in B$, $w := z \in C$,

iff ($z \in A$ AND $z \in B$) OR ($z \in A$ AND $z \in C$)
iff $z \in (A \cap B)$ OR $z \in (A \cap C)$
iff $z \in (A \cap B) \cup (A \cap C)$

Questions?

Functions

A function assigns an element of one set, called the *domain*, to an element of another set, called the *codomain* s.t. for every element in the domain, there is at most one element in the codomain.

If A is the domain and B is the codomain of function f, then $f : A \to B$.

If $a \in A$, and $b \in B$, and $f(a) = b$, we say the function f maps a to b, b is the value of f at argument a, b is the image of a, a is the preimage of b.

$A = \{a, b, c, \ldots z\}$, $B = \{1, 2, 3, \ldots 26\}$, then we can define a function $f : A \to B$ such that $f(a) = 1$, $f(b) = 2$. f thus assigns a number to each letter in the alphabet.

Consider $f : \mathbb{R} \to \mathbb{R}$ s.t. for $x \in \mathbb{R}$, $f(x) = x^2$. $f(2.5) = 6.25 \in \mathbb{R}$.

A function cannot assign different elements in the codomain to the same element in the domain. For example, if $f(a) = 1$ and $f(a) = 2$, the f is not a function.

Functions

A function that assigns a value to every element in the domain is called a *total* function, while one that does not necessarily do so is called a *partial* function.

For $x \in \mathbb{R}$, $f(x) = 1/x^2$ is a partial function because no value is assigned to $x = 0$, since $1/0$ is undefined.

Q: Consider $f : \mathbb{R}_+ \to \mathbb{R}$ such that $f(x) = x$. Is f a function? Ans: Yes

Q: For $x \in [-1, 1], y \in \mathbb{R}$, consider $g(x) = y$ s.t. $x^2 + y^2 = 1$. Is g a function? Ans: No

Q: For $x \in \{-1, 1\}, y \in \mathbb{R}$, consider $g(x) = y$ s.t. $x^2 + y^2 = 1$. Is g a function? Ans: Yes

Functions

We can also define a function with a set as the argument. For a set $S \in D$, $f(S) := \{x | \forall s \in S, x = f(s)\}$.

$A = \{a, b, c, \ldots z\}$, $B = \{1, 2, 3, \ldots 26\}$. $f : A \to B$ such that $f(a) = 1$, $f(b) = 2, \ldots$. $f(\{e, f, z\}) = \{5, 6, 26\}$.

If D is the domain of f, then range$(f) := f(D) = f(\text{domain}(f))$.

Q: If $f : \mathbb{N} \to \mathbb{R}$, and $f(x) = x^2$. What is the domain and codomain of f? What is the range?

Ans: $\mathbb{N}$, $\mathbb{R}$, $\{0, 1, 4, 9, \ldots\}$

Q: Consider $f : \{0, 1\}^5 \to \mathbb{N}$ s.t. $f(x)$ counts the length of a left to right search of the bits in the binary string x until a 1 appears. $f(01000) = 2$.

What is $f(00001)$, $f(00000)$? Is f a total function? Ans: 5, undefined, No

Surjective Functions

Surjective functions: $f : A \to B$ is a surjective function iff for every $b \in B$, there exists an $a \in A$ s.t. $f(a) = b$. $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x + 1$ is a surjective function.

For surjective functions, $|\#\text{arrows}| \geq |B|$.

Since each element of A is assigned at most one value, and some need not be assigned a value at all, $|\#\text{arrows}| \leq |A|$.

Hence, if f is a surjective function, then $|A| \geq |B|$.

$A = \{a, b, c, \ldots z, \alpha, \beta, \gamma, \ldots\}$, $B = \{1, 2, 3, \ldots 26\}$. $f : A \to B$ such that $f(a) = 1$, $f(b) = 2, \ldots$. f does not assign any value to the Greek letters. For every number in B, there is a letter in A. Hence, f is surjective, and $|A| > |B|$.

Injective & Bijective Functions

Injective functions: $f : A \rightarrow B$ is an injective function iff $\forall a \in A$, there is a *unique* $b \in B$ s.t. $f(a) = b$. If f is injective and $f(a) = f(b)$, then it implies that $a = b$.

Hence, $|\#\text{arrows}| = |A| \leq |B|$. Hence, if f is a injective function, then $|A| \leq |B|$.

$A = \{a, b, c, \ldots z\}$, $B = \{1, 2, 3, \ldots 26, 27, \ldots 100\}$. $f : A \rightarrow B$ such that $f(a) = 1$, $f(b) = 2, \ldots$. No element in A is assigned values $27, 28, \ldots$, and for every letter in A, there is a unique number in B. Hence, f is injective, and $|A| < |B|$.

Bijective functions: f is a bijective function iff it is both surjective and injective, implying that $|A| = |B|$.

$A = \{a, b, c, \ldots z\}$, $B = \{1, 2, 3, \ldots 26\}$. $f : A \rightarrow B$ such that $f(a) = 1$, $f(b) = 2, \ldots$. Every element in A is assigned a unique value in B and for every element in B, there is a value in A that is mapped to it. f is bijective, and $|A| = |B|$.

Functions

Converse of the previous statements is also true.

- If $|A| \geq |B|$, then it's always possible to define a surjective function $f : A \to B$.
- If $|A| \leq |B|$, then it's always possible to define a injective function $f : A \to B$.
- If $|A| = |B|$, then it's always possible to define a bijective function $f : A \to B$.

Q: Recall that the Cartesian product of two sets $S = \{s_1, s_2, \dots, s_m\}$, $T = \{t_1, t_2, \dots, t_n\}$ is $S \times T := \{(s,t) | s \in S, t \in T\}$. Construct a bijective function $f : (S \times T) \to \{1, \dots nm\}$, and prove that $|S \times T| = nm$.

Ans: $f(s_1, t_1) = 1$, $f(s_1, t_n) = n$, $f(s_2, t_1) = n + 1$, and so on. $f(s_i, t_j) = n(i-1) + j$. Since f is bijective, $|S \times T| = |\{1, \dots nm\}| = nm$.

Questions?

Sequences

Examples: (a, b, a), (1,3,4), (4,3,1)

An element can appear twice. E.g. $(a, a, b) \neq (a, b)$.

The order of the elements does matter. E.g. $(a, b) \neq (b, a)$.

Q: What is the size of $(1, 2, 2, 3)$? What is the size of $\{1, 2, 2, 3\}$? Ans: 4, 3.

Sets and Sequences: The Cartesian product of sets $S \times T \times U$ is a set consisting of all sequences where the first component is drawn from S, the second component is drawn from T and the third from U. $S \times T \times U = \{(s, t, u) | s \in S, t \in T, u \in U\}$.

Q: For set $S = \{0, 1\}$, $S^3 = S \times S \times S$. Enumerate S^3. What is $|S^3|$?

Ans: $S^3 = \{(0, 0, 0), (0, 0, 1) \ldots (1, 1, 1)\}$, $|S^3| = 8$

Counting Sets - Example

Suppose we want to buy 10 donuts. There are 5 donut varieties – chocolate, lemon-filled, sugar, glazed, plain. Let A be the set of ways to select the 10 donuts. Each element of A is a potential selection. For example, 4 chocolate, 3 lemon, 0 sugar, 2 glazed and 1 plain.

Let's map each way to a string as follows: $\underbrace{0000}_{\text{chocolate}}\ \underbrace{000}_{\text{lemon}}\ \underbrace{\quad}_{\text{sugar}}\ \underbrace{00}_{\text{glazed}}\ \underbrace{0}_{\text{plain}}$.

Lets fix the ordering – chocolate, lemon, sugar, glazed and plain, and abstract this out further to get the sequence: $0000\,1\,000\,1\,1\,00\,1\,0$.

Hence, each way of choosing donuts is mapped to a binary sequence of length 14 with exactly 4 ones. Now, let B be all 14-bit sequences with exactly 4 ones. An element of B is 11110000000000.

Q: The above sequence corresponds to what donut order? Ans: All plain donuts.

For every way to select donuts, we have an equivalent sequence in B. And every sequence in B implies a unique way to select donuts. Hence, the above mapping from $A \to B$ is a bijective function.