CMPT 210: Probability and Computation

Lecture 9

Sharan Vaswani

June 7, 2022

Logistics

- Assignment 1 is marked and the marks are up on Coursys. Average $= 149/170$ and Median $= 157/170$.
- Collect your marked Assignment 1 from TASC-1 9203 between 10.30 am - 12 pm.
- If you emailed your assignment for some reason, please email the TA – Yasaman for your marked assignment.
- Assignment 2 is out: `https://vaswanis.github.io/210-S22/A2.pdf`
  Due Friday 17 June in class.
- For A2, you can use your late-submission and submit on Tuesday 21 June in class.
- To help you prepare for the midterm the solutions will be released on 21 June after class, meaning that no submissions will be allowed after that.
- If you have questions about either assignment or the marking, post it on Piazza: `https://piazza.com/sfu.ca/summer2022/cmpt210/home`

Conditional Probability - Recap

Recall that for events E and F such that $\Pr[F] \neq 0$, $\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$.

For the complement E^c,$\Pr[E^c|F] = 1 - \Pr[E|F]$.

Proof: Since $E \cup E^c = \mathcal{S}$, for an event F such that $\Pr[F] \neq 0$,

$$(E \cup E^c) \cap F = (E \cap F) \cup (E^c \cap F) = \mathcal{S} \cap F = F$$
$$\implies \Pr[E \cap F] + \Pr[E^c \cap F] = \Pr[F] \implies \frac{\Pr[E^c \cap F]}{\Pr[F]} = 1 - \frac{\Pr[E \cap F]}{\Pr[F]}$$
$$\implies \Pr[E^c|F] = 1 - \Pr[E|F]$$

Generalization to multiple events

For events E_1, E_2, E_3, $\Pr[E_1 \cap E_2 \cap E_3] = \Pr[E_1] \Pr[E_2|E_1] \Pr[E_3|E_1 \cap E_2]$.

Proof: By the rule of conditional probability,

$$\Pr[E_1 \cap E_2 \cap E_3] = \Pr[E_1] \Pr[E_2 \cap E_3|E_1] = \Pr[E_1] \Pr[E_2|E_1] \Pr[E_3|E_1 \cap E_2]$$

We can order the events to compute $\Pr[E_1 \cap E_2 \cap E_3]$ more easily. For example,

$$\Pr[E_1 \cap E_2 \cap E_3] = \Pr[E_2] \Pr[E_3|E_2] \Pr[E_1|E_2 \cap E_3]$$

Law of Total Probability and Bayes Rule - Recap

For events E and F such that $\Pr[E] \neq 0$ and $\Pr[F] \neq 0$,

$$\Pr[F|E] = \frac{\Pr[E|F]\Pr[F]}{\Pr[E]} \qquad \text{(Bayes Rule)}$$

For events E and F,

$$\Pr[E] = \Pr[E|F]\Pr[F] + \Pr[E|F^c]\Pr[F^c] \qquad \text{(Law of total probability)}$$

Combining the above equations,

$$\Pr[F|E] = \frac{\Pr[E|F]\ \Pr[F]}{\Pr[E|F]\Pr[F] + \Pr[E|F^c]\Pr[F^c]}$$

Generalization to multiple events

For disjoint events E_1, E_2, E_3 such that $E_1 \cup E_2 \cup E_3 = \mathcal{S}$ and $E_1 \cap E_2 \cap E_3 = \{\}$ i.e. events E_1, E_2 and E_3 form a partition, for any event A,

$$
\begin{aligned}
A &= (A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) && \text{(Since } E_1 \cup E_2 \cup E_3 = \mathcal{S}\text{)} \\
\implies \Pr[A] &= \Pr[A \cap E_1] + \Pr[A \cap E_2] + \Pr[A \cap E_3] && \text{(By union-rule for disjoint events)} \\
\implies \Pr[A] &= \Pr[A|E_1]\Pr[E_1] + \Pr[A|E_2]\Pr[E_2] + \Pr[A|E_3]\Pr[E_3] \\
& && \text{(By definition of conditional probability)}
\end{aligned}
$$

Similarly, we can obtain the Bayes rule for 3 events,

$$
\Pr[E_1|A] = \frac{\Pr[A|E_1]\ \Pr[E_1]}{\Pr[A|E_1]\Pr[E_1] + \Pr[A|E_2]\Pr[E_2] + \Pr[A|E_3]\Pr[E_3]}
$$

Questions?

Total Probability - Examples

Q: An insurance company believes that people can be divided into two classes — those that are accident prone and those that are not. Their statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability 0.4, whereas this probability decreases to 0.2 for a non-accident-prone person. If we assume that 30% of the population is accident prone, what is the probability that a new policy holder will have an accident within a year of purchasing a policy?

Let A = event that a new policy holder will have an accident within a year of purchasing a policy. Let B = event that the new policy holder is accident prone. We know that $\Pr[B] = 0.3$, $\Pr[A|B] = 0.4$, $\Pr[A|B^c] = 0.2$. By the law of total probability,
$\Pr[A] = \Pr[A|B]\Pr[B] + \Pr[A|B^c]\Pr[B^c] = (0.4)(0.3) + (0.2)(0.7) = 0.26$.

Q: Suppose that a new policy holder has an accident within a year of purchasing their policy. What is the probability that they are accident prone?

Compute $\Pr[B|A] = \frac{\Pr[A|B]\ \Pr[B]}{\Pr[A]} = \frac{0.12}{0.26} = 0.4615$.

Total Probability Examples

Q: At a certain stage of a criminal investigation, the inspector in charge is 60% convinced of the guilt of a certain suspect. Suppose now that a new piece of evidence that shows that the criminal has a certain characteristic (such as left-handedness, baldness, brown hair, etc.) is uncovered. If 20% of the general population possesses this characteristic, how certain of the guilt of the suspect should the inspector now be if it turns out that the suspect is among this group?

Let G be the event that the suspect is guilty. Let C be the event that the suspect possesses the characteristic found at the crime scene. We wish to compute $\Pr[G|C]$.

We know that $\Pr[G] = 0.6$, $\Pr[C|G] = 1$, $\Pr[C|G^c] = 0.2$.
$\Pr[C] = \Pr[C|G]\Pr[G] + \Pr[C|G^c]\Pr[G^c] = (1)(0.6) + (0.2)(0.4) = 0.68$

$\Pr[G|C] = \frac{\Pr[G]\ \Pr[C|G]}{\Pr[C]} = \frac{0.6}{0.68} = 0.882.$

Hence, the additional evidence has corroborated the inspector's theory and increased the probability of guilt.

Total Probability - Examples

Alice is taking a probability class and at the end of each week she can be either up-to-date or she may have fallen behind. If she is up-to-date in a given week, the probability that she will be up-to-date (or behind) in the next week is 0.8 (or 0.2, respectively). If she is behind in a given week, the probability that she will be up-to-date (or behind) in the next week is 0.6 (or 0.4, respectively). Alice is (by default) up-to-date when she starts the class. What is the probability that she is up-to-date after three weeks?

Let U_i and B_i be the events that Alice is up-to-date or behind respectively after i weeks. Since Alice starts the class up-to-date, $\Pr[U_1] = 0.8$ and $\Pr[B_1] = 0.2$. We also know that $\Pr[U_2|U_1] = 0.8$, $\Pr[U_3|U_2] = 0.8$ and $\Pr[B_2|U_1] = 0.2$, $\Pr[B_3|U_2] = 0.2$. Similarly, $\Pr[U_2|B_1] = 0.6$, $\Pr[U_3|B_2] = 0.6$ and $\Pr[B_2|B_1] = 0.4$, $\Pr[B_3|B_2] = 0.4$.

We wish to compute $\Pr[U_3]$. By the law of total probability,
$\Pr[U_3] = \Pr[U_3|U_2]\Pr[U_2] + \Pr[U_3|B_2]\Pr[B_2]$ and
$\Pr[U_2] = \Pr[U_2|U_1]\Pr[U_1] + \Pr[U_2|B_1]\Pr[B_1]$.

Hence, $\Pr[U_2] = (0.8)(0.8) + (0.6)(0.2) = 0.76$, and $\Pr[U_3] = (0.8)(0.76) + (0.6)(0.24) = 0.752$.

Simpson's Paradox

In 1973, there was a lawsuit against a university with the claim that a male candidate is more likely to be admitted to the university than a female.

Let us consider a simplified case – there are two departments, EE and CS, and men and women apply to the program of their choice. Let us define the following events: A is the event that the candidate is admitted to the program of their choice, F_E is the event that the candidate is a woman applying to EE, F_C is the event that the candidate is a woman applying to CS. Similarly, we can define M_E and M_C. Assumption: Candidates are either men or women, and that no candidate is allowed to be part of both EE and CS.

Lawsuit claim: Male candidate is more likely to be admitted to the university than a female i.e. $\Pr[A|M_E \cup M_C] > \Pr[A|F_E \cup F_C]$.

University response: In any given department, a male applicant is less likely to be admitted than a female i.e. $\Pr[A|F_E] > \Pr[A|M_E]$ and $\Pr[A|F_C] > \Pr[A|M_C]$.

Simpson's Paradox: Both the above statements can be simultaneously true.

Simpson's Paradox

CS	2 men admitted out of 5 candidates	40%
	50 women admitted out of 100 candidates	50%
EE	70 men admitted out of 100 candidates	70%
	4 women admitted out of 5 candidates	80%
Overall	72 men admitted, 105 candidates	$\approx$ 69%
	54 women admitted, 105 candidates	$\approx$ 51%

In the above example, $\Pr[A|F_E] = 0.8 > 0.7 = \Pr[A|M_E]$ and $\Pr[A|F_C] = 0.5 > 0.4 = \Pr[A|M_C]$. $\Pr[A|F_E \cup F_C] \approx 0.51$. Similarly, $\Pr[A|M_E \cup M_C] \approx 0.69$.

In general, Simpson's Paradox occurs when multiple small groups of data all exhibit a similar trend, but that trend reverses when those groups are aggregated.

Questions?

Back to throwing dice - Independent Events

Suppose we throw two standard dice one after the other. What is the probability that we get two 6's in a row?

E = We get a 6 in the second throw. F = We get a 6 in the first throw. $E \cap F$ = we get two 6's in a row. We are computing $\Pr[E \cap F]$. $\Pr[E] = \Pr[F] = \frac{1}{6}$.

$\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]} \implies \Pr[E \cap F] = \Pr[E|F] \Pr[F]$.

Since the two dice are *independent*, knowing that we got a 6 in the first throw does not change the probability that we will get a 6 in the second throw. Hence, $\Pr[E|F] = \Pr[E]$ (conditioning does not change the probability of the event).

Hence, $\Pr[E \cap F] = \Pr[E|F] \Pr[F] = \Pr[E] \Pr[F] = \frac{1}{6} \frac{1}{6} = \frac{1}{36}$.

Independent Events

Events E and F are said to be independent, if knowledge that F has occurred does not change the probability that E occurs. Formally,

$$\Pr[E \cap F] = \Pr[E] \Pr[F]$$

Q: I toss two independent, fair coins. What is the probability that I get the HT sequence?

Define E to be the event that I get a heads in the first toss, and F be the event that I get a tails in the second toss. Since the two coins are independent, events E and F are also independent. $\Pr[E \cap F] = \Pr[E] \Pr[F] = \frac{1}{2} \frac{1}{2} = \frac{1}{4}$.

Q: I randomly choose a number from $\{1, 2, \ldots, 10\}$. E is the event that the number I picked is a prime. F is the event that the number I picked is odd. Are E and F independent?

$\Pr[E] = \frac{2}{5}$, $\Pr[F] = \frac{1}{2}$, $\Pr[E \cap F] = \frac{3}{10}$. $\Pr[E \cap F] \neq \Pr[E] \Pr[F]$. Another way: $\Pr[E|F] = \frac{3}{5}$ and $\Pr[E] = \frac{2}{5}$, and hence $\Pr[E|F] \neq \Pr[E]$. Conditioning on F tell us that prime number cannot be 2, so it changes the probability of E.

Independent Events - Example

Q: We have a machine that has 2 independent components. The machine breaks if each of its 2 components break. Suppose each component can break with probability p, what is the probability that the machine does not break?

Let E_1 = Event that the first component breaks, E_2 = Event that the second component breaks. M = Event that the machine breaks = $E_1 \cap E_2$.

$\Pr[M] = \Pr[E_1 \cap E_2]$. Since the two components are independent, E_1 and E_2 are independent, meaning that $Pr[E_1 \cap E_2] = Pr[E_1] \Pr[E_2] = p^2$.

Probability that the machine does not break = $\Pr[M^c] = 1 - \Pr[M] = 1 - p^2$.

Independent Events - Example

Q: We have a new machine that breaks if either of its 2 components break. Suppose each component can break with probability p, what is the probability that the machine breaks?

For this machine, let M' be the event that it breaks. In this case, $\Pr[M'] = \Pr[E_1 \cup E_2]$.

Incorrect: By the union rule for mutually exclusive events, $\Pr[E_1 \cup E_2] = \Pr[E_1] + \Pr[E_2] = 2p$.

Mistake: *Independence does not imply mutual exclusivity* and we can not use the union rule for mutually exclusive events. Independence implies that for any two events E and F, $\Pr[E \cap F] = \Pr[E]\Pr[F]$, while mutual exclusivity requires that $\Pr[E \cap F] = 0$.

Correct way 1:

$$\begin{aligned}\Pr[E_1 \cup E_2] &= \Pr[E_1] + \Pr[E_2] - \Pr[E_1 \cap E_2] && \text{(By the union rule)} \\ &= \Pr[E_1] + \Pr[E_2] - \Pr[E_1]\Pr[E_2] = 2p - p^2 && \text{(Since } E_1 \text{ and } E_2 \text{ are independent.)}\end{aligned}$$

Independent Events - Example

Q: We have a new machine that breaks if either of its 2 components break. Suppose each component can break with probability p, what is the probability that the machine breaks?

Correct way 2:

$$\Pr[E_1 \cup E_2] = 1 - \Pr[(E_1 \cup E_2)^c] = 1 - \Pr[(E_1^c \cap E_2^c]$$

(Complement of union of sets is equal to the intersection of the complements of sets)

$$= 1 - \Pr[E_1^c] \Pr[E_2^c] = 1 - (1-p)^2 = 2p - p^2$$

(If E_1 and E_2 are independent, so are E_1^c and E_2^c (Proof on the next slide))

This implies that for the first machine, the probability of failure is p^2 while for the second one, it is $2p - p^2$. Since $p \leq 1$, $p^2 \leq 2p - p^2$, meaning that the first machine fails less often.

Independent Events - Example

Q: Prove that if E_1 and E_2 are independent, so are E_1^c and E_2^c

$$\begin{aligned}
\Pr[(E_1)^c \cap (E_2)^c] &= \Pr[(E_1 \cup E_2)^c] = 1 - \Pr[E_1 \cup E_2] = 1 - \Pr[E_1] - \Pr[E_2] + \Pr[E_1 \cap E_2] \\
&\hspace{20em}\text{(By the union rule)} \\
&= 1 - \Pr[E_1] - \Pr[E_2] + \Pr[E_1]\Pr[E_2] \quad \text{(Since } E_1 \text{ and } E_2 \text{ are independent)} \\
&= (1 - \Pr[E_1])(1 - \Pr[E_2]) = \Pr[E_1^c]\Pr[E_2^c]
\end{aligned}$$