CMPT 210: Probability and Computation

Lecture 8

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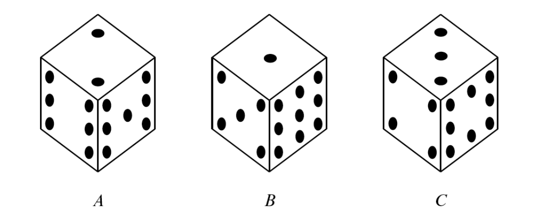
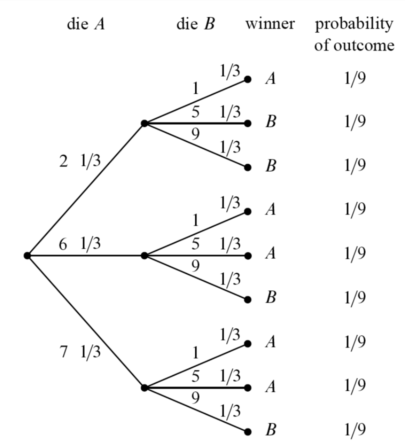
Conditional Probability - Examples

Let us play a game with three strange dice shown in the figure. Each player selects one die and rolls it once. The player with the lower value pays the other player \$100. We can pick a die first, after which the other player can pick one of the other two.



Q: Suppose we choose die B because it has a 9, and the other player selects die A. What is the probability that we will win?

Conditional Probability - Examples



Identify Outcomes: Each leaf is an outcome and $\mathcal{S} = \{(2,1),(2,5),(2,9),(6,1),(6,5),(6,9),(7,1),(7,5),(7,9)\}$.

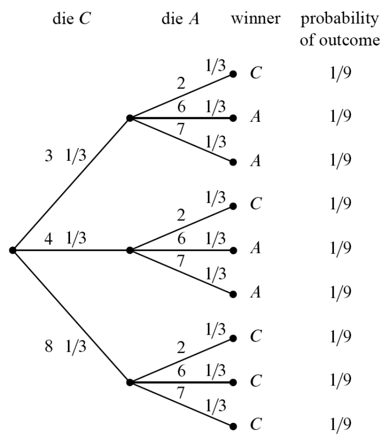
Identify Event: $E = \{(2,5),(2,9),(6,9),(7,9)\}$.

Compute probabilities: $\Pr[\text{Dice 1 is 6}] = \frac{1}{3}$.
$\Pr[(6,5)] = \Pr[\text{Dice 2 is 5} \ \cap \ \text{Dice 1 is 6}] =$
$\Pr[\text{Dice 2 is 5} \mid \text{Dice 1 is 6}] \ \Pr[\text{Dice 1 is 6}] = \frac{1}{3}\frac{1}{3} = \frac{1}{9}$.

$\Pr[E] = \Pr[(2,5)] + \Pr[(2,9)] + \Pr[(6,9)] + \Pr[(7,9)] = \frac{4}{9}$.
Meaning that there is less than 50% chance of winning.

Conditional Probability - Examples

Q: We get another chance – this time we know that die A is good (since we lost to it previously), we choose die A and the other player chooses die C. What is our probability of winning?



Now, $E = \{(3,6),(3,7),(4,6),(4,7)\}$ and hence $\Pr[E] = \frac{4}{9}$. Meaning that there is less than 50% chance of winning.

Conditional Probability - Examples

We get yet another chance, and this time we choose die C, because we reason that die A is better than B, and C is better than A.

By similar reasoning, we can construct a tree diagram to show that unfortunately, the probability that we win is again $\frac{4}{9}$.

So we conclude that,

- A beats B with probability $\frac{5}{9}$ (first game).
- C beats A with probability $\frac{5}{9}$ (second game).
- B beats C with probability $\frac{5}{9}$ (third game).

Since A will beat B more often than not, and B will beat C more often than not, it seems like A ought to beat C more often than not, that is, the "beats more often" relation ought to be transitive. But this intuitive idea is false: whatever die we pick, the second player can pick one of the others and be likely to win. So picking first is actually a disadvantage!

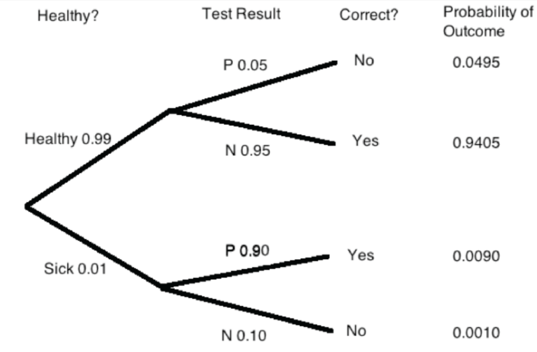
Conditional Probability - Examples

A test for detecting cancer has the following accuracy – (i) If a person has cancer, there is a 10% chance that the test will say that the person does not have it. This is called a "false negative" and (ii) If a person does not have cancer, there is a 5% chance that the test will say that the person does have it. This is called a "false positive". For patients that have no family history of cancer, the incidence of cancer is 1%. Person X does not have any family history of cancer, but is detected to have cancer. What is the probability that the Person X does have cancer?

Conditional Probability - Examples

$\mathcal{S} = \{(Healthy, Positive), (Healthy, Negative), (Sick, Positive), (Sick, Negative)\}$.

A is the event that Person X has cancer. B is the event that the test is positive.



$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{\Pr[\{(S,P)\}]}{\Pr[\{(S,P),(H,P)\}]} = \frac{0.0090}{0.0090+0.0495} \approx 15.4\%.$$

Questions?

Bayes Rule

For events E and F if $\Pr[E] \neq 0$ and $\Pr[F] \neq 0$, then,

$$\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]} \quad ; \quad \Pr[F|E] = \frac{\Pr[F \cap E]}{\Pr[E]}$$

$$\implies \Pr[E \cap F] = \Pr[E|F]\Pr[F] \quad ; \quad \Pr[F \cap E] = \Pr[F|E]\Pr[E]$$

$$\implies \Pr[E|F]\Pr[F] = \Pr[F|E]\Pr[E]$$

$$\implies \Pr[F|E] = \frac{\Pr[E|F]\Pr[F]}{\Pr[E]} \qquad \text{(Bayes Rule)}$$

Allows us to compute $\Pr[F|E]$ using $\Pr[E|F]$. Later in the course, we will see an application of the Bayes rule to machine learning.

Law of Total Probability and Bayes rule

For events E and F,

$$E = (E \cap F) \cup (E \cap F^c)$$

$$\implies \Pr[E] = \Pr[(E \cap F) \cup (E \cap F^c)] = \Pr[E \cap F] + \Pr[E \cap F^c]$$

(By union-rule for disjoint events)

$$\Pr[E] = \Pr[E|F]\Pr[F] + \Pr[E|F^c]\Pr[F^c]$$

(By definition of conditional probability)

$$\Pr[F|E] = \frac{\Pr[F \cap E]}{\Pr[E]} = \frac{\Pr[E|F]\ \Pr[F]}{\Pr[E]}$$

(By definition of conditional probability)

$$\Pr[F|E] = \frac{\Pr[E|F]\ \Pr[F]}{\Pr[E|F]\Pr[F] + \Pr[E|F^c]\Pr[F^c]}$$

(By law of total probability)

Total Probability - Examples

Q: In answering a question on a multiple-choice test, a student either knows the answer or she guesses. Let p be the probability that she knows the answer and $1 - p$ the probability that she guesses. Assume that a student who guesses at the answer will be correct with probability $\frac{1}{m}$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that she answered it correctly?

Let C be the event that the student answers the question correctly. Let K be the event that the student knows the answer. We wish to compute $\Pr[K|C]$.

We know that $\Pr[K] = p$ and $\Pr[C|K^c] = 1/m$, $\Pr[C|K] = 1$. Hence,
$\Pr[C] = \Pr[C|K]\Pr[K] + \Pr[C|K^c]\Pr[K^c] = (1)(p) + \frac{1}{m}(1 - p)$.

$\Pr[K|C] = \frac{\Pr[C|K]\ \Pr[K]}{\Pr[C]} = \frac{mp}{1+(m-1)p}$.