CMPT 210: Probability and Computation

Lecture 7

Sharan Vaswani

May 31, 2022

Recap - Conditional Probability

For events E and F, we wish to compute $\Pr[E|F]$, the probability of event E conditioned on F.

Approach 1: With conditioning, F can be interpreted as the *new sample space* such that for $\omega \notin F$, $\Pr[\omega|F] = 0$.

Example: For computing Pr(we get a 6|the outcome is even), the new sample space is $F = \{2, 4, 6\}$ and the resulting probability space is uniform. $\Pr[\{\text{even number}\}] = \frac{1}{3}$ and $\Pr[\{\text{odd number}\}] = 0$.

Approach 2: $\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$.

Example: $E \cap F = \{6\}$. $\Pr[E \cap F] = \frac{1}{6}$. $\Pr[F] = \Pr[\{2\}] + \Pr[\{4\}] + \Pr[\{6\}] = \frac{1}{2}$. Hence, $\frac{\Pr[E \cap F]}{\Pr[F]} = \frac{1/6}{1/2} = \frac{1}{3}$.

Conditional Probability - Examples

Q: The organization that Jones works for is running a father–son dinner for those employees having at least one son. Each of these employees is invited to attend along with his youngest son. If Jones is known to have two children, what is the conditional probability that they are both boys given that he is invited to the dinner?

Sample space is the pair of genders of Jones' younger and older child. Hence, $\mathcal{S} = \{(b,b),(b,g),(g,b),(g,g)\}$.

The event that we care about is Jones has both boys. Hence, $E = \{(b,b)\}$

Additional information that we are conditioning on is that Jones is invited to the dinner meaning that he has at least one son. Hence, $F = \{(b,b),(b,g),(g,b)\}$.

Hence, $E \cap F = \{(b,b)\}$, $\Pr[E \cap F] = \frac{|E \cap F|}{|\mathcal{S}|} = \frac{1}{4}$. $\Pr[F] = \frac{|F|}{|\mathcal{S}|} = \frac{3}{4}$.

$\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]} = \frac{1/4}{3/4} = \frac{1}{3}$.

Conditional Probability - Examples

Q: Ms. Perez figures that there is a 30 percent chance that her company will set up a branch office in Phoenix. If it does, she is 60 percent certain that she will be made manager of this new operation. What is the probability that Perez will be an office manager in the Phoenix branch?

E = Perez will be a branch office manager; F = her company will set up a branch office in Phoenix; $E \cap F$ = Perez will be an office manager in the Phoenix branch.

From the question, we know that $\Pr[F] = 0.3$, $\Pr[E|F] = 0.6$. Hence,
$\Pr[E \cap F] = \Pr[E] \Pr[E|F] = 0.3 \times 0.6 = 0.18$.

Conditional Probability Examples

Q: Suppose we have a bowl containing 6 white and 5 black balls. We randomly draw a ball. What is the probability that we draw a black ball Ans: $\frac{5}{11}$

Q: We randomly draw two balls, one after the other (without putting the first back). What is the probability that we (i) draw a black ball followed by a white ball (ii) draw a white ball followed by a black ball (iii) we get one black ball and one white ball (iv) both black (v) both white?

B1 = Draw black first, W1 = Draw white first. B2 = Black second, W2 = White second.

(i) $\Pr[B1] = \frac{5}{11}$. $\Pr[W2|B1] = \frac{6}{10}$. Hence, $\Pr[B1 \cap W2] = \Pr[B1] \Pr[W2|B1] = \frac{30}{110}$.

(ii) $\Pr[W1] = \frac{6}{11}$. $\Pr[B2|W1] = \frac{5}{10}$. Hence, $\Pr[W1 \cap B2] = \Pr[W1] \Pr[B2|W1] = \frac{30}{110}$.

(iii) $G = (B1 \cap W2) \cup (W1 \cap B2)$. Events $B1 \cap W2$ and $B2 \cap W1$ are mutually exclusive. By the union rule for mutually exclusive events, $\Pr[G] = \Pr[B1 \cap W2] + \Pr[W1 \cap B2] = \frac{60}{110}$.

(iv) $\Pr[B1 \cap B2] = \Pr[B1] \Pr[B2|B1] = \frac{20}{110}$.

(v) $\Pr[W1 \cap W2] = \Pr[W1] \Pr[W2|W1] = \frac{30}{110}$.

Conditional Probability Examples

Q: Two teams A and B are asked to separately design a new product within a month. From past experience we know that, (a) The probability that team A is successful is 2/3, (b) The probability that team B is successful is 1/2, (c) The probability that at least one team is successful is 3/4. Assuming that exactly one successful design is produced, what is the probability that it was designed by team B.

Let SS be the event that both teams are successful, SF be the event that team A succeeded but team B failed, FS be the event that team B succeeded but team A failed and FF be the event that both teams failed. Hence, $\mathcal{S} = \{(s,s),(s,f),(f,s),(f,f)\}$.

Since exactly one successful design is produced, we know that exactly one of the teams succeeded. Hence, we wish to compute $\Pr[\{FS\}|\{FS \cup SF\}]$.

$\Pr[SS \cup SF] = \frac{2}{3}$, $\Pr[SS \cup FS] = \frac{1}{2}$, $\Pr[SS \cup SF \cup SS] = 3/4$. Since these are mutually exclusive,

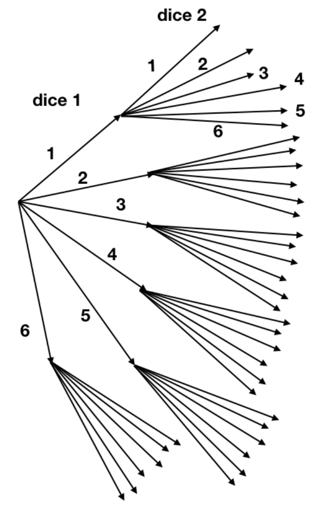
$$\Pr[SS] + \Pr[SF] = \frac{2}{3} \quad \Pr[SS] + \Pr[FS] = \frac{1}{2} \quad \Pr[SS] + \Pr[SF] + \Pr[FS] = \frac{3}{4}$$

Solving these, $\Pr[\{FS\}|\{FS \cup SF\}] = \frac{3/4-2/3}{(3/4-2/3)+(3/4-1/2)} = \frac{1/12}{1/12+1/4} = \frac{1}{4}$.

Questions?

Back to throwing dice - Tree Diagram

Suppose we throw two standard dice one after the other. What is the probability that we get two 6's in a row?



Identify Outcomes: Each leaf is an outcome and $\mathcal{S} = \{(1,1),(1,2),(1,3),\ldots(6,6)\}$.

Identify Event: $E = \{(6,6)\}$.

Compute probabilities: $\Pr[\text{Dice 1 is 6}] = \frac{1}{6}$.
$\Pr[(6,3)] = \Pr[\text{Dice 2 is 3} \;\cap\; \text{Dice 1 is 6}] =$
$\Pr[\text{Dice 2 is 3} \mid \text{Dice 1 is 6}]\ \Pr[\text{Dice 1 is 6}] = \frac{1}{6}\frac{1}{6} = \frac{1}{36}$.

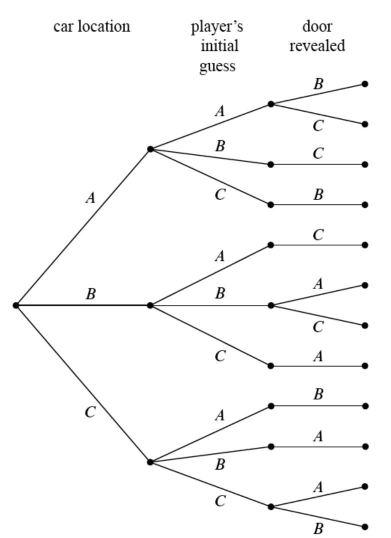
$\Pr[E] = \Pr[\text{dice 1 is 6 } \cap \text{dice 2 is 6 }] = \frac{1}{36}$.

Monty Hall Problem

Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say A, and the host, who knows what's behind the doors, opens another door, say C, which has a goat. He says to you, "Do you want to pick door B?" Is it to your advantage to switch your choice of doors?

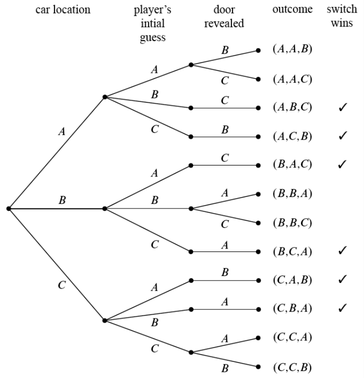
- The car is equally likely to be hidden behind each of the three doors.
- The player is equally likely to pick each of the three doors, regardless of the car's location.
- After the player picks a door, the host must open a different door with a goat behind it and offer the player the choice of staying with the original door or switching.
- If the host has a choice of which door to open, then he is equally likely to select each of them.

Tree Diagram for the Monty Hall Problem - Identify Outcomes



$\mathcal{S} = \{(A, A, B), (A, A, C), (A, B, C), (A, C, B), \ldots\}.$
$E_1 =$ Prize is behind door C $=$
$\{(C, A, B), (C, B, A), (C, C, A), (C, C, B)\}$

Tree Diagram for the Monty Hall Problem - Identify Event

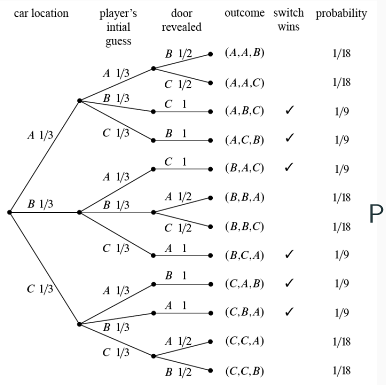


$E = \text{Switching wins} = \{(A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B), (C, B, A)\}$

$\Pr[(A, A)] = \Pr[\text{Car is at A} \cap \text{Player picks A}] = \Pr[\text{Player picks A} \mid \text{Car is at A}] \Pr[\text{Car is at A}] = \frac{1}{3}\frac{1}{3} = \frac{1}{9}$.

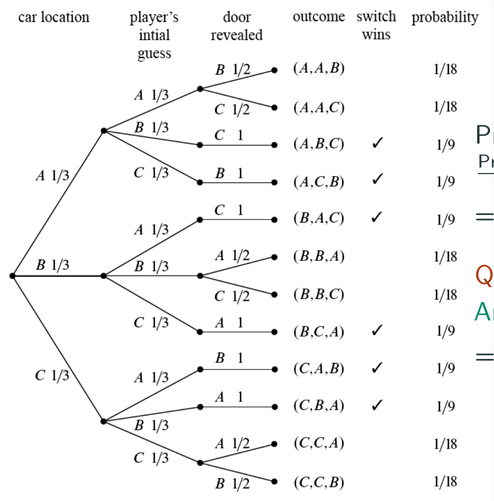
$\Pr[(A, A, B)] = \Pr[\text{Door B is revealed} \cap \text{AA}] = \Pr[\text{Door B is revealed} \mid \text{AA}] \Pr[\text{AA}] = \frac{1}{2}\frac{1}{9} = \frac{1}{18}$.

Tree Diagram for the Monty Hall Problem - Compute Probabilities



$$\Pr[E] = \Pr[(A, B, C)] + \Pr[(A, B, C)] + \Pr[(A, B, C)] + \Pr[(A, B, C)] + \ldots = \frac{1}{9} + \frac{1}{9} + \ldots = \frac{2}{3}.$$

Monty Hall Problem and Conditional Probability



$$\Pr[\text{win by switching} \mid \text{pick A and door B is opened}] = \frac{\Pr[\text{win by switching} \cap \text{pick A and door B is opened}]}{\Pr[\text{pick A and door B is opened}]}$$

$$= \frac{\Pr[(C,A,B)]}{\Pr[\{(A,A,B),(C,A,B)\}]} = \frac{1/9}{1/9+1/18} = \frac{2}{3}.$$

Q: $\Pr[\text{win by switching} \mid \text{pick A and door C is opened}]$?

Ans: $\dfrac{\Pr[\text{win by switching} \cap \text{pick A and door C is opened}]}{\Pr[\text{pick A and door C is opened}]}$

$$= \frac{\Pr[(C,A,C)]}{\Pr[\{(A,A,C),(C,A,C)\}]} = \frac{1/9}{1/9+1/18} = \frac{2}{3}.$$

Questions?

Conditional Probability - Examples

In a best-of-three series, the local hockey team wins the first game with probability $\frac{1}{2}$. In subsequent games, their probability of winning is determined by the outcome of the previous game. If the team won the previous game, then they are invigorated by victory and win the current game with probability $\frac{2}{3}$. If they lost the previous game, then they are demoralized by defeat and win the current game with probability only $\frac{1}{3}$. What is the probability that the local team wins the series, given that they win the first game?

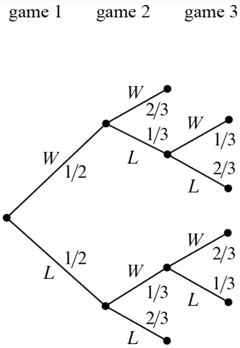
Conditional Probability - Examples

Sample space: $\mathcal{S} = \{(W,W),(W,L,W),(W,L,L),(L,W,W),(L,W,L),(L,L)\}$.

Events: $T = \{(W,W),(W,L,W),(L,W,W)\}$, $F = \{(W,W),(W,L,W),(W,L,L)\}$.

$$
\begin{aligned}
\Pr[T|F] &= \frac{\Pr[T \cap F]}{\Pr[F]} \\
&= \frac{\Pr[\{(W,W),(W,L,W)\}]}{\Pr[\{(W,W),(W,L,W),(W,L,L)\}]} \\
&= \frac{\Pr[\{(W,W)\} + \Pr[\{(W,L,W)\}]}{\Pr[\{(W,W)\} + \Pr[\{(W,L,W)\} + \Pr[\{(W,L,L)\}]]} \\
&= \frac{1/3 + 1/18}{1/3 + 1/18 + 1/9} = \frac{7}{9}
\end{aligned}
$$

Conditional Probability - Examples



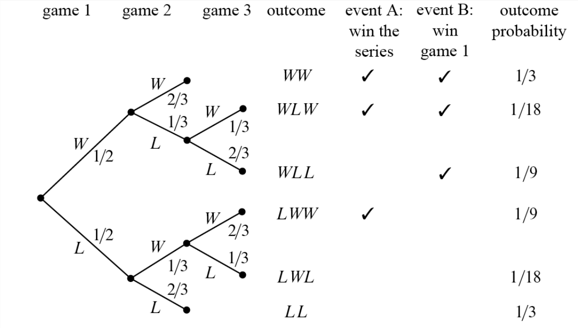
outcome	event A: win the series	event B: win game 1	outcome probability
WW	✓	✓	1/3
WLW	✓	✓	1/18
WLL		✓	1/9
LWW	✓		1/9
LWL			1/18
LL			1/3

Q: What is the probability that the team wins the series if they lose Game 1? Ans: $\frac{1/9}{1/9+1/18+1/3} = \frac{2}{9}$

Q: What is the probability that the team wins the series? Ans: $\frac{1}{2}$

Q: What is the probability that the series goes to Game 3? Ans: $\frac{1}{3}$

Conditional Probability - Examples



Q: What is the probability that the team won their first game given that they won the series?

Recall that $T = \{(W, W), (W, L, W), (L, W, W)\}$, $F = \{(W, W), (W, L, W), (W, L, L)\}$. We wish to compute $\Pr[F|T] = \frac{\Pr[F \cap T]}{\Pr[T]} = \frac{\Pr[\{(W,W),(W,L,W)\}]}{\Pr[\{(W,W),(W,L,W),(L,W,W)\}]} = \frac{1/3+1/18}{1/3+1/18+1/9} = \frac{7}{9}$.