CMPT 210: Probability and Computation

Lecture 6

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Recap - Axioms of Probability

Sample (outcome) space $\mathcal{S}$: Nonempty (countable) set of possible outcomes. Example: When we threw one dice, the sample space is $\{1, 2, 3, 4, 5, 6\}$.

Outcome $\omega \in \mathcal{S}$: Possible "thing" that can happen. Example: When we threw one dice, a possible outcome is $\omega = 1$.

Event E: Any subset of the sample space. Example: When we threw one dice, a possible event is $E = \{6\}$ (first example) or $E = \{3, 6\}$ (second example).

Probability function on a sample space $\mathcal{S}$ is a total function $\Pr : \mathcal{S} \rightarrow [0, 1]$. For any $\omega \in \mathcal{S}$,

$$0 \leq Pr[\omega] \leq 1 \quad ; \quad \sum_{\omega \in \mathcal{S}} \Pr[\omega] = 1 \quad ; \quad \Pr[E] = \sum_{\omega \in E} \Pr[\omega]$$

Recap - Probability rules

Union: For mutually exclusive events $E_1, E_2, \ldots, E_n$,
$\Pr[E_1 \cup E_2 \cup \ldots E_n] = \Pr[E_1] + \Pr[E_2] + \ldots + \Pr[E_n]$.

Complement rule: $\Pr[E] = 1 - \Pr[E^c]$

Inclusion-Exclusion rule: For any two events E, F, $\Pr[E \cup F] = \Pr[E] + \Pr[F] - \Pr[E \cap F]$.

Union Bound: For any events $E_1, E_2, E_3, \ldots E_n$, $\Pr[E_1 \cup E_2 \cup E_3 \ldots \cup E_n] \leq \sum_{i=1}^{n} \Pr[E_i]$.

Uniform probability space: A probability space is said to be uniform if $\Pr[\omega]$ is the same for every outcome $\omega \in \mathcal{S}$. In this case, $\Pr[E] = \frac{|E|}{|\mathcal{S}|}$.

Probability - Examples

Q: Let us consider random permutations of the letters (i) ABBA (ii) ABBA'. What is the probability that the third letter is B?

Ans: **(i)** $|\mathcal{S}| = \frac{4!}{2!\,2!} = 6$. $|E| = \frac{3!}{2!1!} = 3$. $\Pr[E] = \frac{1}{2}$.
(ii) $|\mathcal{S}| = \frac{4!}{2!\,1!\,1!} = 12$. $|E| = \frac{3!}{1!1!} = 6$. $\Pr[E] = \frac{1}{2}$.

Questions?

Conditional Probability

Conditioning is revising probabilities based on partial information (an event).

For example, suppose we throw a "standard" dice, what is the probability of getting a 6 if we are told that the outcome is even? Compute Pr(we get a 6|the outcome is even) (Probability of getting a 6 *given* that the outcome is even)/(Probability of getting a 6 *conditioned on the event* that the outcome is even).

Sample space: $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$, Event: $E = \{6\}$. Additional information: Event $F = \{2, 4, 6\}$ has happened. With conditioning, F can be interpreted as the *new sample space*.

Since each outcome in $F = \{2, 4, 6\}$ is still equally likely, the new probability space is still a uniform probability space. Hence, conditioned on the event that the outcome is even, $\sum_{\omega \in F} \Pr[\omega] = 1$ and $\Pr[\{\text{even number}\}] = \frac{1}{3}$ and $\Pr[\{\text{odd number}\}] = 0$.

Q: What is the probability of getting either a 3 or 6 if we are told that the outcome is even? Ans: $E = \{3, 6\}$, $F = \{2, 4, 6\}$. $\Pr[\{3\}] = 0$. $\Pr[\{6\}] = 1/3$. Hence, $\Pr[E] = \Pr[\{3\}] + \Pr[\{6\}] = \frac{1}{3}$.

Conditional Probability

For two events E and F, we wish to compute the probability of event E conditioned on F i.e. event F has happened/constrained to happen.

By conditioning on F, the only outcomes we care about are in F i.e. for $\omega \notin F$, $\Pr[\omega|F] = 0$.

Since we want to compute the probability that event E happens, we care about the outcomes that are in E. Hence, the outcomes we care about lie in both E *and* F, meaning that $\omega \in E \cap F$.
$\implies \Pr[E|F] \propto \sum_{\omega \in (E \cap F)} \Pr[\omega]$. By definition of proportionality, for some constant $c > 0$, $\Pr[E|F] = c \sum_{\omega \in (E \cap F)} \Pr[\omega]$.

We know that $\Pr[F|F] = 1$ (probability of event F given that F has happened). Hence, $\Pr[F|F] = 1 = c \sum_{\omega \in F} \Pr[\omega] \implies c = \frac{1}{\sum_{\omega \in F} \Pr[\omega]}$.

Substituting the value of c,

$$\Pr[E|F] = \frac{\sum_{\omega \in (E \cap F)} \Pr[\omega]}{\sum_{\omega \in F} \Pr[\omega]} = \frac{\Pr[E \cap F]}{\Pr[F]}, \text{where } \Pr[F] \neq 0.$$

Back to throwing dice

Suppose we throw a "standard" dice, what is the probability of getting a 6 if we are told that the outcome is even?

Sample space: $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$, Event: $E = \{6\}$. We are conditioning on $F = \{2, 4, 6\}$.

Hence, $\Pr(\text{we get a } 6 | \text{the outcome is even}) = \Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$.

$E \cap F = \{6\}$. $\Pr[E \cap F] = \frac{1}{6}$. $\Pr[F] = \Pr[\{2\}] + \Pr[\{4\}] + \Pr[\{6\}] = \frac{1}{2}$. Hence, $\frac{\Pr[E \cap F]}{\Pr[F]} = \frac{1/6}{1/2} = \frac{1}{3}$.

Q: What is the probability of getting either a 3 or 6 if we are told that the outcome is even? Ans: $E = \{3, 6\}$, $F = \{2, 4, 6\}$. $E \cap F = \{6\}$. $\Pr[E] = \Pr[\{6\}] = \frac{1}{6}$. $\Pr[\{F\}] = \frac{1}{2}$. Hence, $\Pr[E|F] = \frac{1}{3}$.

Questions?

Conditional Probability - Examples

Q: Suppose we select a card at random from a standard deck of 52 cards. What is the probability of getting:

- A spade conditioned on the event that I picked the red color Ans: 0
- A spade facecard conditioned on the event that I picked the black color Ans: $\frac{3}{26}$
- A black card conditioned on the event that I picked a spade facecard Ans: 1
- The queen of hearts given that I picked a queen Ans: $\frac{1}{4}$
- An ace given that I picked a spade Ans: $\frac{1}{13}$