CMPT 210: Probability and Computation

Lecture 23

Sharan Vaswani

August 2, 2022

Continuous Random Variables

We have studied random variables that can take on discrete values – number of heads when tossing a coin, the number on a dice or the number of attempts to hit the bullsye in a dart game.

We have used these discrete distributions for designing randomized algorithms for verifying matrix multiplication, finding the maximum cut in graphs and sorting. We have also seen applications to polling, A/B testing and binary classification in machine learning.

It is often more natural to model quantities as continuous random variables, for example, the amount of time it takes to transmit a message over a noisy channel or study the distribution of income in a population.

Continuous random variables are often used in distributed computing and for regression – fitting a model that can effectively explain the collected data.

Continuous Random Variables

Discrete random variables can take on specific values in an interval. For example, if $X \sim \text{Uniform}\{v_1, v_2, \ldots, v_n\}$, X can take on values from the set $\{v_1, v_2, \ldots, v_n\}$. If $X \sim \text{Bin}(n, p)$, X can take on values in the set $\{0, , 1, \ldots, n\}$.

Continuous random variable: A r.v. that can take on all possible values in a specified interval.

For example, if $R \sim \text{Uniform}[0, 1]$, the r.v. R can be equal to any number in the $[0, 1]$ interval – for example, 0.01, 2/3 or 0.9.

Continuous uniform distribution: A r.v. R follows a continuous uniform distribution if it has equal probability of taking on any value in the $[a, b]$ interval (for $a \leq b$). It is denoted as $R \sim \text{Uniform}[a, b]$.

An important special case is $R \sim \text{Uniform}[0, 1]$ i.e. R is uniform on the unit interval.

Continuous Uniform Distribution as a limit of the Discrete distribution

The continuous uniform distribution can be interpreted as a limit of the discrete distribution. To see this, let us discretize the $[0, 1]$ interval into N bins for large N (E.g. $N \approx 2^{64}$).

Define r.v. X s.t. $X \sim \text{Uniform}\{0, 1/N, 2/N, \ldots, 1\}$. For all $x \in \text{Range}(X)$, $\Pr[X = x] = \frac{1}{N}$.

Let us compute $\Pr[X \leq 0.3]$.

$$X \leq 0.3 = (X = 0) \cup \left(X = \frac{1}{N}\right) \cup \ldots \cup \frac{\lfloor 0.3N \rfloor}{N}$$

$$\implies \Pr[X \leq 0.3] = \Pr[X = 0] + \Pr\left[X = \frac{1}{N}\right] + \ldots + \Pr\left[X = \frac{\lfloor 0.3N \rfloor}{N}\right] = \frac{\lfloor 0.3N \rfloor + 1}{N}$$

By definition of the floor function, $\lfloor 0.3N \rfloor \in [0.3N - 1, 0.3N]$. Hence,

$$\Pr[X \leq 0.3] \in \left[0.3, 0.3 + \frac{1}{N}\right]$$

As $N \to \infty$, $\frac{1}{N} \to 0$, and hence, $\Pr[X \leq 0.3] \approx 0.3$.

Continuous Uniform Distribution as a limit of the Discrete distribution

Similarly, we can show that as $N \to \infty$, $\Pr[X \geq 0.4] = \left[0.6 - \frac{1}{N}, 0.6\right] \approx 0.6$.

More generally, for $u, v \in [0, 1]$ and $u \leq v$, $\Pr[u \leq X \leq v] \in \left[v - u - \frac{1}{N}, v - u + \frac{1}{N}\right] \approx v - u$.

The continuous uniform distribution can be recovered from the discrete uniform distribution as $N \to \infty$. If $R \sim \text{Uniform}[0, 1]$ then, for $u \leq v$ and $u, v \in [0, 1]$,

$$\Pr[u \leq R \leq v] = v - u$$

For a small quantity du, if $v = u + du$, then, $\Pr[u \leq R \leq u + du] = du$.

Q: If $R \sim \text{Uniform}[0, 1]$, compute $\Pr[R = 0.3]$?

Consider the discrete distribution $X \sim \text{Uniform}\{0, 1/N, 2/N, \ldots, 1\}$, if $0.3 \notin \text{Range}(X)$, then, $\Pr[X = 0.3] = 0$. If $0.3 \in \text{Range}(X)$, then, $\Pr[X = 0.3] = \frac{1}{N}$.
$\Pr[R = 0.3] = \lim_{N \to \infty} \Pr[X = 0.3] = \lim_{N \to \infty} \frac{1}{N} = 0$. Hence, for all $x \in [0, 1]$, $\Pr[R = x] = 0$.

For continuous distributions, the probability that R is equal to a specific value is zero!

Continuous Uniform Distribution

Consider the more general $R \sim \text{Uniform}[a, b]$. As before, R can be interpreted as the limit of $X \sim \text{Uniform}\{a, a+1/N, a+2/N, \ldots, b\}$.

By using the same reasoning on the transformed r.v. equal to $Y = \frac{X-a}{b-a}$, one can show that

$$\Pr[R \leq u] = \frac{u-a}{b-a}$$
$$\Pr[u \leq R \leq v] = \frac{v-u}{b-a}$$
$$\Pr[u \leq R \leq u + du] = \frac{du}{b-a}$$

PDF and CDF

Probability Density Function: The PDF for a continuous r.v. R is denoted by $f_R : \mathbb{R} \to \mathbb{R}_+$. For all u, $f_R(u) \geq 0$ and for all $a \leq b$, it satisfies $\Pr[R \in [a, b]] = \int_a^b f_R(u)\, du$.

Since $\Pr[R \in (-\infty, \infty)] = 1 \implies \int_{-\infty}^{\infty} f_R(u)\, du = 1$.

Cumulative Distribution Function: The CDF for a continuous r.v. R is denoted by $F_R : \mathbb{R} \to [0, 1]$ and defined as $F_R(v) := \Pr[R \leq v] = \int_{-\infty}^{v} f_R(u)\, du$. Hence, $\lim_{v \to -\infty} F_R(v) = 0$ and $\lim_{v \to \infty} F_R(v) = 1$.

PDF and CDF: For any continuous r.v. R, $\frac{dF_R(v)}{dv} = \frac{d \int_{-\infty}^{v} f_R(u)\, du}{dv} = f_R(v)$.

Example: For $R \sim \text{Uniform}[a, b]$, $f_R(u) = \frac{1}{b-a}$ for all $u \in [a, b]$, else $f_R(u) = 0$.

$$\int_{-\infty}^{\infty} f_R(u)\, du = \int_a^b f_R(u)\, du = \frac{1}{b-a} \int_a^b 1\, du = 1.$$

$$F_R(v) = \Pr[R \leq v] = \int_{-\infty}^{v} f_R(u)\, du = \int_a^v f_R(u)\, du = \frac{1}{b-a} \int_a^v 1\, du = \frac{v-a}{b-a}.$$

Verifying the relation between the PDF and the CDF, $\frac{dF_R(v)}{dv} = \frac{1}{b-a} = f_R(v)$.

Expectation and Variance

Q: If $R \sim \text{Uniform}[0,1]$, what is $f_R(u)$, $F_R(u)$? Ans: For all $u \in [0,1]$, $f_R(u) = 1$ and $F_R(u) = u$.

For $R \sim \text{Uniform}[0,1]$, let us compute $\mathbb{E}[R]$ by using the discrete approximation.

$$\mathbb{E}[X] = \sum_{x=0}^{N} \frac{x}{N} \Pr\left[X = \frac{x}{N}\right] = \frac{1}{N^2} \sum_{x=0}^{N} x = \frac{N(N+1)}{2N^2} = \frac{1}{2} + \frac{1}{2N}$$

Hence, as $N \to \infty$, $\mathbb{E}[X] \approx \frac{1}{2}$ and hence $\mathbb{E}[R] = \frac{1}{2}$.

Q: Using the discrete uniform distribution as a proxy, compute $\text{Var}[R]$ for $R \sim \text{Uniform}[0,1]$.

Let us first compute $\mathbb{E}[X^2]$.

$$\mathbb{E}[X^2] = \sum_{x=0}^{N} \frac{x^2}{N^2} \Pr\left[X = \frac{x}{N}\right] = \frac{1}{N^3} \sum_{x=0}^{N} x^2 = \frac{N\,(N+1)\,(2N+1)}{6N^3} = \left(1 + \frac{1}{N}\right)\left(\frac{1}{3} + \frac{1}{6N}\right)$$

Hence, as $N \to \infty$, $\mathbb{E}[X^2] \approx \frac{1}{3}$ and hence $\mathbb{E}[R^2] = \frac{1}{3}$.

Hence, $\text{Var}[R] = \mathbb{E}[R^2] - (\mathbb{E}[R])^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$.

Expectation and Variance

Recall the definition of expectation for $X \sim \text{Uniform}\{0, 1/N, \ldots, 1\}$.

$$\mathbb{E}[X] = \sum_{x=0}^{N} \frac{x}{N} \Pr\left[X = \frac{x}{N}\right] = \sum_{x=0}^{N} \frac{x}{N} \frac{1}{N}$$

As $N \to \infty$, by definition of an integral,

$$\mathbb{E}[R] \approx \int_0^1 u \, du = \int_0^1 u \, f_R(u) \, du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2} \qquad \text{(Since } f_R(u) = 1 \text{ for all } u \in [0,1].\text{)}$$

Expectation: For a continuous r.v. R with PDF equal to f_R, $\mathbb{E}[R] = \int_{-\infty}^{\infty} u \, f_R(u) \, du$.

Q: If $R \sim \text{Uniform}[a, b]$, compute $\mathbb{E}[R]$

$$\mathbb{E}[R] = \int_{-\infty}^{\infty} u \, f_R(u) \, du = \int_a^b \frac{u}{b-a} \, du = \frac{1}{b-a} \frac{u^2}{2}\Big|_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}.$$

If $a = 0$ and $b = 1$, $\mathbb{E}[R] = \frac{1}{2}$ and we recover the result of Slide 7.

Expectation and Variance

Similar to the discrete case, we can generalize the definition of expectation to compute $\mathbb{E}[g(R)]$ where g is an arbitrary function of R.

$$\mathbb{E}[g(R)] = \int_{-\infty}^{\infty} g(u)\, f_R(u)\, du$$

Q: If $R \sim \text{Uniform}[a, b]$, compute $\text{Var}[R]$

In order to compute the variance, we need to compute the second moment i.e. $g(R) = R^2$.

$$\mathbb{E}[R^2] = \int_{-\infty}^{\infty} u^2\, f_R(u)\, du = \int_a^b u^2\, \frac{1}{b-a}\, du = \frac{1}{b-a}\, \frac{u^3}{3}\Big|_a^b = \frac{b^3 - a^3}{3\,(b-a)} = \frac{a^2 + ab + b^2}{3}$$

$$\text{Var}[R] = \mathbb{E}[R^2] - (\mathbb{E}[R])^2 = \frac{a^2 + ab + b^2}{3} - \frac{(b+a)^2}{4} = \frac{(b-a)^2}{12}$$

If $a = 0$ and $b = 1$, $\text{Var}[R] = \frac{1}{12}$ and we recover the result of Slide 7.

Summary

The distribution of a continuous r.v. R is completely specified by its PDF $f_R : \mathbb{R} \to \mathbb{R}_+$ and CDF $F_R : \mathbb{R} \to [0, 1]$.

Probability Density Function: For all u, $f_R(u) \geq 0$ and for all $a \leq b$, it satisfies $\Pr[R \in [a, b]] = \int_a^b f_R(u)\, du$. $\int_{-\infty}^{\infty} f_R(u)\, du = 1$.

Cumulative Distribution Function: For all u, $F_R(u) := \Pr[R \leq u] = \int_{-\infty}^{u} f_R(u)\, du$ and satisfies: $\lim_{u \to -\infty} F_R(u) = 0$ and $\lim_{u \to \infty} F_R(u) = 1$.

PDF and CDF: For any continuous r.v. R, $\frac{dF_R(v)}{dv} = \frac{d \int_{-\infty}^{v} f_R(u)\, du}{dv} = f_R(v)$.

Expectation and Variance: For a continuous r.v. R, $\mathbb{E}[R] = \int_{-\infty}^{\infty} u\, f_R(u)\, du$ and $\text{Var}[R] = (\int_{-\infty}^{\infty} u^2\, f_R(u)\, du) - (\int_{-\infty}^{\infty} u\, f_R(u)\, du)^2$.

Continuous uniform distribution: If $R \sim \text{Uniform}[a, b]$, for all $u \in [a, b]$, $f_R(u) = \frac{1}{b-a}$ and $f_R(u) = 0$ if $u \notin [a, b]$. $\forall u \in [a, b]$, $F_R(u) = \frac{u-a}{b-a}$. $F_R(u) = 0$ if $u < a$ and $F_R(u) = 1$ if $u > b$.

Expectation and Variance for the continuous uniform distribution: If $R \sim \text{Uniform}[a, b]$, $\mathbb{E}[R] = \frac{b+a}{2}$ and $\text{Var}[R] = \frac{(b-a)^2}{12}$.

Continuous Random Variables – Example

Q: Suppose that X is a continuous random variable whose probability density function is given by:

$$f_X(x) = C\,(4x - 2x^2) \qquad \text{(for } 0 < x < 2\text{)}$$
$$= 0 \qquad \text{(otherwise)}$$

(i) Determine C (ii) Compute $\Pr[X > 1]$ (iii) Compute $\mathbb{E}[X]$ and (iv) $\mathrm{Var}[X]$.

Ans: (i) Since the distribution has to integrate to 1, $C \int_0^2 (4x - 2x^2)\,dx = 1$. Hence, $2x^2\left(1 - \frac{x}{3}\right)\Big|_0^2 = \frac{1}{C} \implies C = \frac{3}{8}$.

(ii) $\Pr[X > 1] = \int_1^2 \frac{3}{8}\,(4x - 2x^2) = \frac{3}{4}\,x^2\left(1 - \frac{x}{3}\right)\Big|_1^2 = \frac{1}{2}$.

(iii) $\mathbb{E}[X] = \int_0^2 x\,\frac{3}{8}\,(4x - 2x^2) = \frac{3}{4}\left(\frac{2x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^2 = 1$.

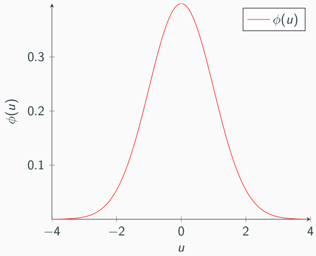
(iv) $\mathbb{E}[X^2] = \frac{3}{4}\left(\frac{2x^4}{4} - \frac{x^5}{5}\right)\Bigg|_0^2 = 1.6$. Hence, $\mathrm{Var}[X] = 1.6 - 1^2 = 0.6$.

Questions?

Standard Normal (Gaussian) Distribution

Random variable R follows the standard normal distribution i.e. $X \sim \mathcal{N}(0,1)$ if

$$f_R(u) = \Phi(u) := \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-u^2}{2}\right)$$



Clearly, $f_R(u) > 0$ for all $u \in \mathbb{R}$. Q: Sanity check: Is f_R a valid PDF i.e. is $\int_{-\infty}^{\infty} f_R(u)\, du = 1$?

Yes! The integral can not be solved by standard techniques – need to reparameterize the problem in terms of the polar coordinates (r, θ) and solve it (you will do it in STAT 271).

Let us derive the mean of the standard normal distribution.

$$\mathbb{E}[R] = \int_{-\infty}^{\infty} u\, f_R(u)\, du = \int_{-\infty}^{\infty} u \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-u^2}{2}\right) du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u \exp\left(\frac{-u^2}{2}\right) du$$

Note that $g(u) := u \exp\left(\frac{-u^2}{2}\right)$ is an odd function i.e $g(-u) = -g(u)$. Hence,

$$\mathbb{E}[R] = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} g(u)\, du + \int_{0}^{\infty} g(u)\, du\right] = \frac{1}{\sqrt{2\pi}} \left[\int_{0}^{\infty} g(-u)\, du + \int_{0}^{\infty} g(u)\, du\right]$$

$$\implies \mathbb{E}[R] = \frac{1}{\sqrt{2\pi}} \left[\int_{0}^{\infty} -g(u)\, du + \int_{0}^{\infty} g(u)\, du\right] = 0.$$

Standard Normal (Gaussian) Distribution

Let us derive the variance of the standard normal distribution.

$$\mathrm{Var}[R] = \mathbb{E}[R^2] - (\mathbb{E}[R])^2 = \mathbb{E}[R^2] = \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 \exp\left(\frac{-z^2}{2}\right) dz$$

Note that $h(z) := z^2 \exp\left(\frac{-z^2}{2}\right)$ is an even function i.e $h(-z) = h(z)$. Hence,

$$\begin{aligned}\mathrm{Var}[R] &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} h(z) + \int_{0}^{\infty} h(z)\right] dz = \frac{1}{\sqrt{2\pi}} \left[\int_{0}^{\infty} h(-z) + \int_{0}^{\infty} h(z)\right] dz \\ &= \sqrt{\frac{2}{\pi}} \left[\int_{0}^{\infty} h(z)\right] dz = \sqrt{\frac{2}{\pi}} \left[\int_{0}^{\infty} z^2 \exp\left(\frac{-z^2}{2}\right)\right] dz\end{aligned}$$

Let us solve this integral using integration by parts: $\int_0^\infty u\, dv = uv - \int_0^\infty v\, du$. In our case, we set $u = z$ and $v = \exp\left(\frac{-z^2}{2}\right)$. Hence, $du = dz$ and $dv = -z \exp\left(\frac{-u^2}{2}\right) dz$ and $\sqrt{\frac{2}{\pi}} \left[\int_0^\infty z^2 \exp\left(\frac{-z^2}{2}\right)\right] dz = \sqrt{\frac{2}{\pi}} \int_\infty^0 u\, dv$.

Standard Normal (Gaussian) Distribution

Recall that we need to solve: $\sqrt{\frac{2}{\pi}} \int_{\infty}^{0} u\, dv$ where $u = z$ and $v = \exp\left(\frac{-z^2}{2}\right)$.

$$\int_{\infty}^{0} u\, dv = z \exp\left(\frac{-z^2}{2}\right)\Bigg|_{\infty}^{0} - \int_{\infty}^{0} \exp\left(\frac{-z^2}{2}\right)\, dz = \int_{0}^{\infty} \exp\left(\frac{-z^2}{2}\right)\, dz$$
$$= \frac{1}{2} \int_{-\infty}^{\infty} \exp\left(\frac{-z^2}{2}\right)\, dz$$

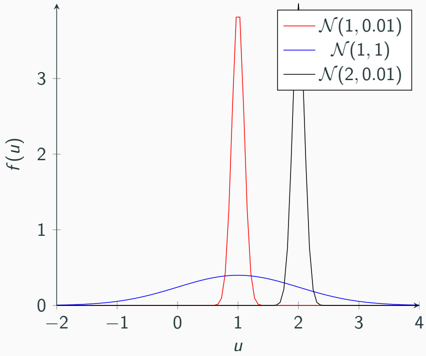
This is the Gaussian PDF upto an $\sqrt{2\pi}$ factor, and hence, $\int_{\infty}^{0} u\, dv = \frac{\sqrt{2\pi}}{2} = \sqrt{\frac{\pi}{2}}$.

Putting everything together, $\mathrm{Var}[R] = \sqrt{\frac{2}{\pi}} \int_{\infty}^{0} u\, dv = 1$. Hence, if $R \sim \mathcal{N}(0, 1)$, then $\mathbb{E}[R] = 0$ and $\mathrm{Var}[R] = 1$.

Normal Distribution

In general, random variable R follows the Normal distribution i.e. $R \sim \mathcal{N}(\mu, \sigma^2)$, if

$$f_R(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$



If $R \sim \mathcal{N}(\mu, \sigma^2)$, $\mathbb{E}[R] = \mu$ and $\text{Var}[R] = \sigma^2$.

Properties of the Normal Distribution

Standardizing a Gaussian r.v: If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$.

As a check, $\mathbb{E}[Z] = \mathbb{E}\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}\mathbb{E}[X-\mu] = \frac{1}{\sigma}(\mathbb{E}[X]-\mu) = 0$.

$\mathrm{Var}[Z] = \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2 = \mathbb{E}[Z^2] = \mathbb{E}\left[\frac{(X-\mu)^2}{\sigma^2}\right] = \frac{1}{\sigma^2}\mathbb{E}[(X-\mu)^2] = \frac{1}{\sigma^2}\mathrm{Var}[X] = \frac{1}{\sigma^2}\sigma^2 = 1$.

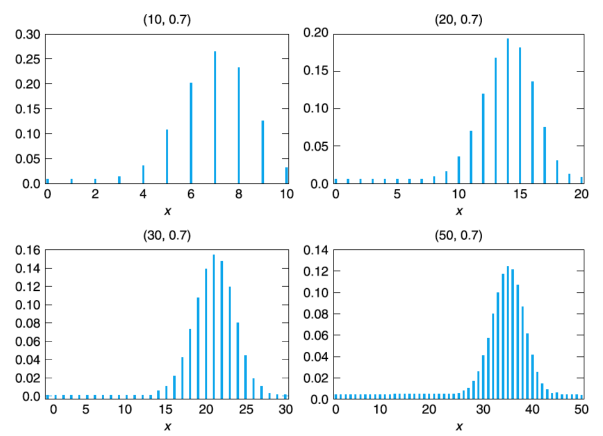
We can prove that all the moments of Z are equal to those of $\mathcal{N}(0,1)$ (by using the moment generating function like in Assignment 4) or by arguing using the CDF (you will do this in STAT 271)

The converse also holds: if $Z \sim \mathcal{N}(0,1)$, then $X = (\mu + \sigma Z) \sim \mathcal{N}(\mu, \sigma^2)$.

Questions?

Normal Distribution as a limit of the Binomial distribution

The normal distribution can be interpreted as a limit of the Binomial distribution $X \sim \mathrm{Bin}(n, p)$ as $n \to \infty$.



Normal Distribution as a limit of the Binomial distribution

Let us derive the Normal distribution as a limit of the Binomial distribution (for large n). If $X \sim \text{Bin}(n, p)$, then for $q := 1 - p$,

$$f_X(x) = \Pr[X = x] = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

Using the Sterling approximation to $n!$ from Lecture 2: For large n, $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

$$\approx \frac{n^n e^{-n}\sqrt{2\pi n}}{x^x e^{-x}\sqrt{2\pi x}\,(n-x)^{n-x} e^{-(n-x)}\sqrt{2\pi(n-x)}} p^x q^{n-x} = \frac{n^n\, p^x q^{n-x}}{x^x\,(n-x)^{n-x}} \sqrt{\frac{n}{2\pi x\,(n-x)}}$$

$$= \left(\frac{np}{x}\right)^x \left(\frac{nq}{n-x}\right)^{n-x} \sqrt{\frac{n}{2\pi x\,(n-x)}}$$

Let us focus on the first two terms $C := \left(\frac{np}{x}\right)^x \left(\frac{nq}{n-x}\right)^{n-x}$ and define $\delta := x - np$. We will only consider x close to the mean np, hence $\delta = x - np$ is small (near zero).

$$\ln(C) = x \ln\left(\frac{np}{x}\right) + (n-x) \ln\left(\frac{nq}{n-x}\right) = -(\delta + np) \ln\left(1 + \frac{\delta}{np}\right) - (nq - \delta) \ln\left(1 - \frac{\delta}{nq}\right)$$

Normal Distribution as a limit of the Binomial distribution

Recall that $\ln(C) = -(\delta + np)\ln\left(1+\frac{\delta}{np}\right) - (nq-\delta)\ln\left(1-\frac{\delta}{nq}\right)$. Using the Taylor series approximation – for $y \approx 0$, $\ln(1+y) \approx y - \frac{y^2}{2}$. Using $y = \delta/np$ for the first term and $y = \frac{-\delta}{nq}$ for the second term (since δ is small and n is large, $y \approx 0$ in both cases).

$$\begin{aligned}
\ln(C) &= -(\delta + np)\left[\frac{\delta}{np} - \frac{\delta^2}{2n^2p^2}\right] - (nq-\delta)\left[\frac{-\delta}{nq} - \frac{\delta^2}{2n^2q^2}\right] \\
&= -\frac{\delta^2}{np} - \delta + \frac{\delta^3}{2n^2p^2} + \frac{\delta^2}{2np} + \delta + \frac{\delta^2}{2nq} - \frac{\delta^2}{nq} - \frac{\delta^3}{2n^2q^2} \\
&= -\frac{-\delta^2}{2np} - \frac{\delta^2}{2nq} + O\left(\frac{\delta^3}{n^2}\right) = -\frac{\delta^2}{2npq} + O\left(\frac{\delta^3}{n^2}\right)
\end{aligned}$$

$$\implies \ln(C) \approx -\frac{\delta^2}{2npq} \implies C = \exp\left(-\frac{\delta^2}{2npq}\right) \qquad \text{(Ignoring the small } O\left(\tfrac{\delta^3}{n^2}\right) \text{ term)}$$

Putting everything together, for large n,

$$f_X(x) \approx \exp\left(-\frac{\delta^2}{2npq}\right)\sqrt{\frac{n}{2\pi x\,(n-x)}}$$

Normal Distribution as a limit of the Binomial distribution

Recall that $f_X(x) \approx \exp\left(-\frac{\delta^2}{2npq}\right)\sqrt{\frac{n}{2\pi x\,(n-x)}}$. Let us simplify the second term,

$$\sqrt{\frac{n}{2\pi x\,(n-x)}} = \sqrt{\frac{np}{x}\,\frac{nq}{n-x}}\sqrt{\frac{1}{2\pi\,npq}}$$

Let us focus on the first two terms. Recall that $\delta = x - np \implies x = np + \delta$ and $n - x = nq - \delta$.

$$\implies \sqrt{\frac{np}{x}\,\frac{nq}{n-x}} = \sqrt{\frac{np}{np+\delta}\,\frac{nq}{nq-\delta}} = \sqrt{\frac{1}{1+\delta/np}}\sqrt{\frac{1}{1-\delta/nq}}$$

Using the Taylor series approximation, for $x \approx 0$, $1/\sqrt{1+x} \approx 1 - \frac{x}{2}$. Since δ is small and n is large, both δ/np and δ/nq are small and we can use the Taylor series approximation.

$$\sqrt{\frac{np}{x}\,\frac{nq}{n-x}} \approx \left(1 - \frac{\delta}{2np}\right)\left(1 + \frac{\delta}{2nq}\right) = 1 + O\left(\frac{\delta}{n}\right) \approx 1.$$

(Ignoring the small $O\left(\frac{\delta}{n}\right)$ term)

Putting everything together, $f_X(x) \approx \exp\left(-\frac{\delta^2}{2npq}\right)\sqrt{\frac{1}{2\pi\,npq}}$

Normal Distribution as a limit of the Binomial distribution

We have seen that for large n, $f_X(x) \approx \exp\left(-\frac{\delta^2}{2npq}\right)\sqrt{\frac{1}{2\pi\, npq}}$. Since $\delta = x - np$,

$$f_X(x) \approx \exp\left(-\frac{(x-np)^2}{2npq}\right)\sqrt{\frac{1}{2\pi\, npq}}$$

Recall that if $X \sim \mathrm{Bin}(n,p)$, $\mu := \mathbb{E}[X] = np$ and $\sigma^2 = \mathrm{Var}[X] = npq$. Hence,

$$f_X(x) \approx \sqrt{\frac{1}{2\pi\,\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

This is exactly the Normal distribution on Slide 12! Hence for large n and x close to the mean np, the Binomial behaves as a Gaussian. In other words, the Gaussian distribution can be interpreted as a limit (for large n) of the Binomial distribution.

Questions?