CMPT 210: Probability and Computation

Lecture 22

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Two-sided Chernoff Bound

Chernoff Bound: Let $T_1, T_2, \ldots, T_n$ be mutually independent r.v's such that $0 \leq T_i \leq 1$ for all i. If $T := \sum_{i=1}^{n} T_i$, for all $c \geq 1$ and $\beta(c) := c\ln(c) - c + 1$,

$$\Pr[T \geq c\mathbb{E}[T]] \leq \exp(-\beta(c)\,\mathbb{E}[T])$$

Two-sided Chernoff Bound: Bounds the probability that the r.v. T takes values (much) smaller or larger than the mean $\mathbb{E}[T]$:

$$\Pr\left[|T - \mathbb{E}[T]| \geq c\mathbb{E}[T]\right] \leq 2\exp\left(\frac{-c^2\mathbb{E}[T]}{3}\right)$$

A/B Testing

Fussbook is redesigning their website to try to make it more appealing to users. In order to see if the new redesigned website actually helps, Fussbook decides to do an experiment.

Given m users of the website, let $\mathcal{A}$ be the set of users who engage with the Fussbook posts (liking, sharing, etc) if *they are shown the old website*. Similarly, $\mathcal{B}$ is the set of users that engage if *they are shown the new website*.

Define f_A (and f_B) to be the fraction of users that engage when shown the old (or new website respectively), i.e. $f_A := \frac{|\mathcal{A}|}{m}$ and $f_B := \frac{|\mathcal{B}|}{m}$. We assume that these fractions correspond to the proportion of users that prefer the old website vs the new.

The improvement on switching to the new website is defined in terms of the **lift** which is equal to $f_B - f_A$. If $f_B - f_A > 0$, then it makes sense to switch to the new website.

A/B Testing

In order to estimate f_A and f_B and see if the new website actually helps, Fussbook decides to run an **A/B** test on m users.

Algorithm A/B Testing

```
function ABTest(m)
X_A = 0, X_B = 0 {Initialize number of users that engaged when shown the old/new website}
for i = 1,2, ..., m do
    X_i ~ Ber(1/2)
    if X_i = 1 then
        Show user i old website.    If user engages, X_A = X_A + 1
    else
        Show user i new website.    If user engages, X_B = X_B + 1
    end if
end for
return  f̂_A = 2X_A/m and f̂_B = 2X_B/m.
```

A/B Testing

Claim: $\hat{f}_A$ and $\hat{f}_B$ are unbiased estimators of f_A and f_B respectively, i.e $\mathbb{E}[\hat{f}_A] = f_A$ and $\mathbb{E}[\hat{f}_B] = f_B$. Note that $X_A = \sum_{i \in \mathcal{A}} X_i$.

$$\mathbb{E}[\hat{f}_A] = \mathbb{E}\left[\frac{2X_A}{m}\right] = \frac{2}{m}\mathbb{E}\left[\sum_{i \in \mathcal{A}} X_i\right] = \frac{2}{m}\left[\sum_{i \in \mathcal{A}} \mathbb{E}[X_i]\right] = \frac{2}{m}\left[\sum_{i \in \mathcal{A}} \Pr[X_i = 1]\right] = \frac{2}{m}\frac{|\mathcal{A}|}{2} = f_A.$$

Similarly, $X_B = \sum_{i \in \mathcal{B}} (1 - X_i)$.

$$\mathbb{E}[\hat{f}_B] = \mathbb{E}\left[\frac{2X_B}{m}\right] = \frac{2}{m}\mathbb{E}\left[\sum_{i \in \mathcal{B}} (1 - X_i)\right] = \frac{2}{m}\left[\sum_{i \in \mathcal{B}} \mathbb{E}[1 - X_i]\right] = \frac{2}{m}\left[\sum_{i \in \mathcal{B}} \Pr[X_i = 0]\right] = \frac{2}{m}\frac{|\mathcal{B}|}{2} = f_B.$$

A/B Testing

Claim: With probability 0.95, the algorithm ABTest(m) estimates the lift to an error equal to 0.1 if $m \approx 10517$.

$$\Pr\left[|\hat{f}_A - f_A| \geq \epsilon\right] = \Pr\left[\left|\frac{2X_A}{m} - f_A\right| \geq \epsilon\right] = \Pr\left[\left|X_A - \frac{m\,f_A}{2}\right| \geq \frac{m\epsilon}{2}\right]$$

$$= \Pr\left[\left|X_A - \frac{m\,f_A}{2}\right| \geq \underbrace{\frac{\epsilon}{f_A}}_{c}\ \frac{mf_A}{2}\right]$$

$$\leq 2\exp\left(-\frac{1}{3}\left(\frac{\epsilon}{f_A}\right)^2\frac{mf_A}{2}\right) \qquad \text{(Two-sided Chernoff Bound)}$$

$$= 2\exp\left(-\frac{\epsilon^2}{6}\frac{m}{f_A}\right) \leq 2\exp\left(-\frac{\epsilon^2\,m}{6}\right)$$

(Since $f_A < 1$ and exp is a monotonically increasing function.)

Similarly, we can prove that, $\Pr\left[|\hat{f}_B - f_B| \geq \epsilon\right] \leq 2\exp\left(-\frac{\epsilon^2\,m}{6}\right)$.

A/B Testing

By the union-bound for two events A and B, $\Pr[A \cup B] \leq \Pr[A] + \Pr[B]$. Hence,

$$\Pr\left[|\hat{f}_A - f_A| \geq \epsilon \cup |\hat{f}_B - f_B| \geq \epsilon\right] \leq 4\exp\left(-\frac{\epsilon^2 m}{6}\right)$$

$$\implies \Pr\left[|\hat{f}_A - f_A| < \epsilon \cap |\hat{f}_B - f_B| < \epsilon\right] \geq 1 - 4\exp\left(-\frac{\epsilon^2 m}{6}\right)$$

$$|\text{estimated lift} - \text{true lift}| = |[\hat{f}_B - \hat{f}_A] - [f_B - f_A]| = |[\hat{f}_B - f_B] + [f_A - \hat{f}_A]|$$

$$\leq |[\hat{f}_B - f_B]| + |[f_A - \hat{f}_A]|$$

(Triangle Inequality: For any constants a, b, $|a + b| \leq |a| + |b|$)

$$\implies \Pr[|\text{estimated lift} - \text{true lift}| \leq 2\epsilon] \geq 1 - 4\exp\left(-\frac{\epsilon^2 m}{6}\right).$$

Proof of Triangle inequality: For any a, b, $ab \leq |a||b|$.

$\implies a^2 + b^2 + 2ab \leq a^2 + b^2 + 2|a||b| = |a|^2 + |b|^2 + 2|a||b|$ (since $\forall x, x^2 = |x|^2$).

$\implies (a+b)^2 = (\,|(a+b)|\,)^2 \leq (|a| + |b|)^2 \implies |a + b| \leq |a| + |b|$.

A/B Testing

In the claim, we want the RHS to be equal to 0.95 and $2\epsilon = 0.1$. Hence,

$$4 \exp\left(-\frac{\epsilon^2 m}{6}\right) = 0.05 \implies m = \frac{6 \ln(80)}{\epsilon^2} = \frac{6 \ln(80)}{(0.05)^2} \approx 10517.$$

Hence, Fussbook can estimate the lift upto an error of 0.1 accurately with probability 0.95 by running the A/B test on $m \approx 10517$ users.

Questions?