CMPT 210: Probability and Computation

Lecture 21

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Recap

Tail inequalities bound the probability that the r.v. takes a value much different from its mean.

Markov's Theorem: If X is a non-negative random variable, then for all $x > 0$,
$\Pr[X \geq x] \leq \frac{\mathbb{E}[X]}{x}$.

Chebyshev's Theorem: For a r.v. X and all $x > 0$, $\Pr[|X - \mathbb{E}[X]| \geq x] \leq \frac{\mathrm{Var}[X]}{x^2}$.

Sums of Random Variables

If we know that the r.v X is (i) non-negative and (ii) $\mathbb{E}[X]$, we can use Markov's Theorem to bound the probability of deviation from the mean.

If we know both (i) $\mathbb{E}[X]$ and (ii) $\text{Var}[X]$, we can use Chebyshev's Theorem to bound the probability of deviation.

In many cases (the voter poll example), we know the distribution of the r.v. (for voter poll, $S_n \sim \text{Bin}(n, p)$) and can obtain tighter bounds on the deviation from the mean.

Chernoff Bound: Let $T_1, T_2, \ldots, T_n$ be mutually independent r.v's such that $0 \leq T_i \leq 1$ for all i. If $T := \sum_{i=1}^{n} T_i$, for all $c \geq 1$ and $\beta(c) := c \ln(c) - c + 1$,

$$\Pr[T \geq c\mathbb{E}[T]] \leq \exp(-\beta(c)\, \mathbb{E}[T])$$

If $T_i \sim \text{Ber}(p)$ and are mutually independent, then $T_i \in \{0, 1\}$ and we can use the Chernoff bound to bound the deviation from the mean for $T \sim \text{Bin}(n, p)$. In general, if $T_i \in [0, 1]$, the Chernoff Bound can be used even if the T_i's have different distributions!

Chernoff Bound – Binomial Distribution

Q: Bound the probability that the number of heads that come up in 1000 independent tosses of a fair coin exceeds the expectation by 20% or more.

Let T_i be the r.v. for the event that coin i comes up heads, and let T denote the total number of heads. Hence, $T = \sum_{i=1}^{1000} T_i$. For all i, $T_i \in \{0, 1\}$ and are mutually independent r.v's. Hence, we can use the Chernoff Bound.

We want to compute the probability that the number of heads is larger than the expectation by 20% meaning that $c = 1.2$ for the Chernoff Bound. Computing $\beta(c) = c\ln(c) - c + 1 \approx 0.0187$. Since the coin is fair, $\mathbb{E}[T] = 1000\,\frac{1}{2} = 500$. Plugging into the Chernoff Bound,

$$\Pr[T \geq c\mathbb{E}[T]] \leq \exp(-\beta(c)\mathbb{E}[T]) \implies \Pr[T \geq 1.2\mathbb{E}[T]] \leq \exp(-(0.0187)\,(500)) \approx 0.0000834.$$

Comparing this to using Chebyshev's inequality,

$$\Pr[T \geq c\mathbb{E}[T]] = \Pr[T - \mathbb{E}[T] \geq (c-1)\mathbb{E}[T]] \leq \Pr[|T - \mathbb{E}[T]| \geq (c-1)\,\mathbb{E}[T]]$$

$$\leq \frac{\mathrm{Var}[T]}{(c-1)^2\,(\mathbb{E}[T])^2} = \frac{1000\,\frac{1}{4}}{(1.2-1)^2(500^2)} = \frac{250}{0.2^2\,500^2} = \frac{250}{10000} = 0.025.$$

Chernoff Bound – Lottery Game

Q: Pick-4 is a lottery game in which you pay \$1 to pick a 4-digit number between 0000 and 9999. If your number comes up in a random drawing, then you win \$5,000. Your chance of winning is 1 in 10000. If 10 million people play, then the expected number of winners is 1000. When there are 1000 winners, the lottery keeps \$5 million of the \$10 million paid for tickets. The lottery operator's nightmare is that the number of winners is much greater – especially at the point where more than 2000 win and the lottery must pay out more than it received. What is the probability that will happen?

Let T_i be an indicator for the event that player i wins. Then $T := \sum_{i=1}^{n} T_i$ is the total number of winners. If we assume that the players' picks and the winning number are random, independent and uniform, then the indicators T_i are independent, as required by the Chernoff bound.

We wish to compute $\Pr[T \geq 2000] = \Pr[T \geq 2\mathbb{E}[T]]$. Hence $c = 2$ and $\beta(c) \approx 0.386$. By the Chernoff bound,

$$\Pr[T \geq 2\mathbb{E}[T]] \leq \exp(-\beta(c)\mathbb{E}[T]) = \exp(-(0.386)\,1000) < \exp(-386) \approx 10^{-168}$$

Questions?

Chernoff Bound – Proof

We want to compute $\Pr[T \geq c\mathbb{E}[T]] = \Pr[f(T) \geq f(c\mathbb{E}[T])]$ where f is a one-one monotonically non-decreasing function. For $c \geq 1$, choosing $f(T) = c^T$ and using Markov's Theorem,

$$\Pr[T \geq c\mathbb{E}[T]] = \Pr[c^T \geq c^{c\mathbb{E}[T]}] \leq \frac{\mathbb{E}[c^T]}{c^{c\mathbb{E}[T]}}$$

$$\leq \frac{\exp((c-1)\,\mathbb{E}[T])}{c^{c\mathbb{E}[T]}} \qquad \text{(To prove next: } \mathbb{E}[c^T] \leq \exp((c-1)\,\mathbb{E}[T]))$$

$$= \frac{\exp((c-1)\,\mathbb{E}[T])}{\exp(\ln(c^{c\mathbb{E}[T]}))} = \frac{\exp((c-1)\,\mathbb{E}[T])}{\exp(c\mathbb{E}[T]\,\ln(c))} = \exp\left(-(c\ln(c) - c + 1)\,\mathbb{E}[T]\right)$$

$$\implies \Pr[T \geq c\mathbb{E}[T]] \leq \exp\left(-\beta(c)\,\mathbb{E}[T]\right)$$

The proof would be done if we prove that $\mathbb{E}[c^T] \leq \exp((c-1)\,\mathbb{E}[T])$ and we do this next.

Chernoff Bound – Proof

Claim: $\mathbb{E}[c^T] \leq \exp((c-1)\,\mathbb{E}[T])$

$$\mathbb{E}[c^T] = \mathbb{E}[c^{\sum_{i=1}^n T_i}] = \mathbb{E}\left[\prod_{i=1}^n c^{T_i}\right] = \prod_{i=1}^n \mathbb{E}[c^{T_i}]$$

(Expectation of product of mutually independent r.v's is equal to the product of the expectation.)

For two variables, the proof of the above statement is in Lecture 15 and can be easily generalized.

$$\leq \prod_{i=1}^n \exp((c-1)\,\mathbb{E}[T_i]) \qquad \text{(To prove next: } \mathbb{E}[c^{T_i}] \leq \exp((c-1)\,\mathbb{E}[T_i])\text{)}$$

$$= \exp\left((c-1)\sum_{i=1}^n \mathbb{E}[T_i]\right) = \exp\left((c-1)\,\mathbb{E}\left[\sum_{i=1}^n T_i\right]\right)$$

(Linearity of Expectation)

$$\implies \mathbb{E}[c^T] \leq \exp((c-1)\mathbb{E}[T])$$

The proof would be done if we prove that $\mathbb{E}[c^{T_i}] \leq \exp((c-1)\,\mathbb{E}[T_i])$ and we do this next.

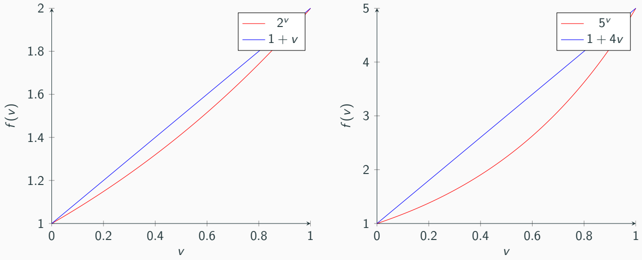
Chernoff Bound – Proof

Claim: $\mathbb{E}[c^{T_i}] \leq \exp((c-1)\,\mathbb{E}[T_i])$

$$\mathbb{E}[c^{T_i}] = \sum_{v \in \text{Range}(T_i)} c^v \Pr[T_i = v] \leq \sum_{v \in \text{Range}(T_i)} (1 + (c-1)v) \Pr[T_i = v]$$

(Since $T_i \in [0,1]$ and $c^v \leq 1 + (c-1)v$ for all $v \in [0,1]$.)

For $c = 2$ and $c = 5$,

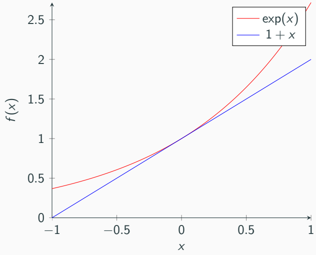


Chernoff Bound – Proof

$$\mathbb{E}[c^{T_i}] \leq \sum_{v \in \text{Range}(T_i)} \Pr[T_i = v] + (c-1) \sum_{v \in \text{Range}(T_i)} v \Pr[T_i = v]$$

$$= 1 + (c-1)\mathbb{E}[T_i] \leq \exp((c-1)\mathbb{E}[T_i]) \qquad (\text{Since } 1 + x \leq \exp(x) \text{ for all } x)$$

$$\implies \mathbb{E}[c^{T_i}] \leq \exp((c-1)\,\mathbb{E}[T_i])$$



Hence we have proved the Chernoff Bound!

Comparing the Bounds

For r.v's $T_1, T_2, \ldots T_n$, if $T_i \in \{0,1\}$ and $\Pr[T_i = 1] = p_i$. Define $T := \sum_{i=1}^{n} T_i$. By linearity of expectation, $\mathbb{E}[T] = \sum_{i=1}^{n} p_i$. For $c \geq 1$,

Markov's Theorem: $\Pr[T \geq c\mathbb{E}[T]] \leq \frac{1}{c}$. Does not require T_i's to be independent.

Chebyshev's Theorem: $\Pr[T \geq c\mathbb{E}[T]] \leq \frac{\text{Var}[T]}{(c-1)^2(\mathbb{E}[T])^2}$. If the T_i's are pairwise independent, by linearity of variance, $\text{Var}[T] = \sum_{i=1}^{n} p_i\,(1 - p_i)$. Hence, $\Pr[T \geq c\mathbb{E}[T]] \leq \frac{\sum_{i=1}^{n} p_i\,(1-p_i)}{(c-1)^2\left(\sum_{i=1}^{n} p_i\right)^2}$. If for all i, $p_i = 1/2$, then, $\Pr[T \geq c\mathbb{E}[T]] \leq \frac{1}{(c-1)^2}$.

Chernoff Bound: If T_i' are mutually independent, then,
$\Pr[T \geq c\mathbb{E}[T]] \leq \exp(-\beta(c)\,\mathbb{E}[T]) = \exp\left(-(c\ln(c) - c + 1)\left(\sum_{i=1}^{n} p_i\right)\right)$. If for all i, $p_i = 1/2$,
$\Pr[T \geq c\mathbb{E}[T]] \leq \exp\left(-\frac{n\,(c\ln(c)-c+1)}{2}\right)$.

Questions?

Randomized Load Balancing

Fussbook is a new social networking site oriented toward unpleasant people. Like all major web services, Fussbook has a load balancing problem: it receives lots of forum posts that computer servers have to process. If any server is assigned more work than it can complete in a given interval, then it is overloaded and system performance suffers. That would be bad because Fussbook users are not a tolerant bunch.

The programmers of Fussbook just randomly assigned posts to computers, and to their surprise the system has not crashed yet.

Fussbook receives 24000 forum posts in every 10-minute interval. Each post is assigned to one of several servers for processing, and each server works sequentially through its assigned tasks. It takes a server an average of $1/4$ second to process a post. No post takes more than 1 second.

This implies that a server could be overloaded when it is assigned more than 600 units of work in a 10-minute interval. On average, for $24000 \times \frac{1}{4} = 6000$ units of work in a 10-minute interval, Fussbook requires at least 10 servers to ensure that no server is overloaded (with perfect load-balancing).

Randomized Load Balancing

Q: There might be random fluctuations in the load or the load-balancing is not perfect. How many servers does Fussbook need to ensure that their servers are not overloaded with high-probability?

Let m be the number of servers that Fussbook needs to use. Recall that a server may be overloaded if the load it is assigned exceeds 600 units. Let us first look at server 1 and define T be the r.v. corresponding to the number of units of work assigned to the first server.

Let T_i be the number of seconds server 1 spends on processing post i. $T_i = 0$ if the task is assigned to a different (not the first server). The maximum amount of time spent on post i is 1-second. Hence, $T_i \in [0, 1]$.

Since there are $n := 24000$ posts in every 10-minute interval, the load (amount of units) assigned to the first server is equal to $T = \sum_{i=1}^{n} T_i$. Server 1 may be overloaded if $T \geq 600$, and hence we want to upper-bound the probability $\Pr[T \geq 600]$.

Since the assignment of a post to a server is independent of the time required to process the post, the T_i r.v's are mutually independent. Hence, we can use the Chernoff bound.

Randomized Load Balancing

We first need to estimate $\mathbb{E}[T]$.

$$\mathbb{E}[T] = \mathbb{E}[\sum_{i=1}^{n} T_i] = \sum_{i=1}^{n} \mathbb{E}[T_i] \qquad \text{(Linearity of expectation)}$$

$$\begin{aligned}
\mathbb{E}[T_i] &= \sum_{i=1}^{n} \mathbb{E}[T_i | \text{server 1 is assigned post } i] \Pr[\text{server 1 is assigned post } i] \\
&\quad + \mathbb{E}[T_i | \text{server 1 is not assigned post } i] \Pr[\text{server 1 is not assigned post } i] \\
&= \frac{1}{4}\frac{1}{m} + (0)(1 - 1/m) = \frac{1}{4m}.
\end{aligned}$$

$$\implies \mathbb{E}[T] = \sum_{i=1}^{n} \frac{1}{4m} = \frac{n}{4m} = \frac{6000}{m}.$$

Randomized Load Balancing

Recall the Chernoff Bound: $\Pr[T \geq c\mathbb{E}[T]] \leq \exp(-\beta(c)\,\mathbb{E}[T])$. In our case, $c\mathbb{E}[T] = 600 \implies c = \frac{m}{10}$. Hence,

$$\Pr[T \geq 600] \leq \exp\left(-\beta\left(\frac{m}{10}\right)\frac{6000}{m}\right)$$

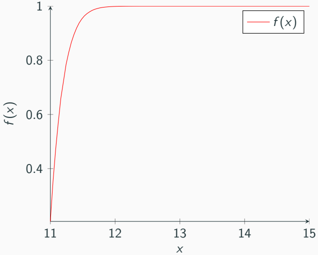
Hence, $\Pr[\text{first server is overloaded}] = \Pr[T \geq 600] \leq \exp\left(-\beta\left(\frac{m}{10}\right)\frac{6000}{m}\right)$.

$$\begin{aligned}
&\Pr[\text{some server is overloaded}] \\
&= \Pr[\text{server 1 is overloaded} \cup \text{server 2 is overloaded} \cup \ldots \cup \text{server m is overloaded}] \\
&\leq \sum_{j=1}^{m} \Pr[\text{server j is overloaded}] && \text{(Union Bound)} \\
&= m \Pr[\text{server 1 is overloaded}] = m \exp\left(-\beta\left(\frac{m}{10}\right)\frac{6000}{m}\right) \\
& && \text{(Since all servers are equivalent)}
\end{aligned}$$

$$\implies \Pr[\text{no server is overloaded}] \geq 1 - m\exp\left(-\beta\left(\frac{m}{10}\right)\frac{6000}{m}\right).$$

Randomized Load Balancing

Plotting Pr[no server is overloaded] as a function of m.



Hence, as $m \geq 12$, the probability that no server gets overloaded tends to 1 and hence none of the Fussbook servers crash!

Questions?