CMPT 210: Probability and Computation

Lecture 2

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Counting sets - using the generalized product rule

Q: Suppose we have p prizes to be handed amongst the set A of n students. What are the number of ways in which we can distribute the prizes? Ans: Consider sequences of length p where element i is the student that receives prize i. The element i can be one of n students. The number of sequences is equal to $|A \times A \times \ldots| = |A|^p = n^p$.

Suppose we have p prizes to be handed amongst the set A of n students. What are the number of ways in which we can distribute the prizes such that each prize goes to a different student i.e. no student receives more than one prize?

Consider sequences of length p. The first entry can be chosen in n ways (the first prize can be given to one of the n students). After the first entry is chosen, since the same student cannot receive the prize, the second entry can be chosen in $n-1$ ways, and so on. Hence, the total number of ways to distribute the prizes $= n \times (n-1) \times \ldots \times (n-(p-1))$.

Generalized product rule: If S is the set of length k sequences such that the first entry can be selected in n_1 ways, after the first entry is chosen, the second one can be chosen in n_2 ways, and so on, then $|S| = n_1 \times n_2 \times \ldots n_k$. If $n_1 = n_2 = \ldots = n_k$, we recover the product rule.

Counting Sets - Example

Q: A dollar bill is defective if some digit appears more than once in the 8-digit serial number. What is the fraction of non-defective bills?

In order to compute the fraction of non-defective bills, we need to compute the quantity $\frac{|\text{serial numbers with all different digits}|}{|\text{possible serial numbers}|}$.

For computing $|\text{possible serial numbers}|$, each digit can be one of 10 numbers. Hence, $|\text{possible serial numbers}| = 10 \times 10 \ldots = 10^8$.

For computing $|\text{serial numbers with all different digits}|$, the first digit can be one of 10 numbers. Once the first digit is chosen, the second one can be chosen in 9 ways, and so on. By the generalized product rule, $|\text{serial numbers with all different digits}| = 10 \times 9 \times \ldots 3 = 1,814,400$.

Fraction of non-defective bills $= \frac{1{,}814{,}400}{10^8} = 1.8144\%$.

Permutations

A permutation of a set S is a sequence of length $|S|$ that contains every element of S exactly once.

Permutations of $\{a, b, c\}$ are $(a, b, c), (a, c, b), (b, c, a), (b, a, c), (c, a, b), (c, b, a)$.

Given a set of size n, what is the total number of permutations?

Considering sequences of length n, the first entry can be chosen in n ways. Since each element can be chosen only once, the second entry can be chosen in $n - 1$ ways, and so on.

By the generalized product rule, the number of permutations $= n \times (n-1) \times \ldots \times 1$.

Factorial: $n! = n \times (n-1) \times \ldots \times 1$. By convention: $0! = 1$.

How big is $n!$? **Stirling approximation**: $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$.

Q: Which is bigger? $n!$ vs $n(n-1)(n+2)\,(n-3)!$? Ans:
$n! = n(n-1)(n-2)(n-3)! < n(n-1)(n+2)(n-3)!$.

Q: In how many ways can we arrange n people in a line? Ans: $n!$

Counting sets - Division rule

k-**to**-1 **function**: Maps exactly k elements of the domain to every element of the codomain.

If $f : A \to B$ is a k-to-1 function, then, $|A| = k|B|$.

Example: E is the set of ears in this room, and P is the set of people. Then f mapping the ears to people is a 2-to-1 function. Hence, $|E| = 2|P|$.

Q: If $f : A \to B$ is a k-to-1 function, and $g : B \to C$ is a m-to-1 function, then what is $|A|/|C|$?

Ans: $|A| = k|B| = km|C|$. Hence $|A|/|C|$ is km.

Q: If $f : A \to B$ is a k-to-1 function, and $g : C \to B$ is a m-to-1 function, then what is $|A|/|C|$?

Ans: $|A| = k|B|$. $|C| = m|B|$. $|A|/|C| = \frac{k}{m}$.

Counting sets - Example

In how many ways can we arrange n people around a round table? Two seatings are considered to be the same *arrangement* if each person sits with the same person on their left in both seatings.

Starting from the head of the table, and going clockwise, each seating has an equivalent sequence. $|\text{seatings}| = \text{number of permutations} = n!$.

n different seatings can result in the same arrangement (by clockwise rotation).

Hence, $f : \text{seatings} \to \text{arrangements}$ is an n-to-1 function. Hence, the $|\text{seatings}| = n\,|\text{arrangements}|$, meaning that the $|\text{arrangements}| = (n-1)!$.

Questions?

Counting subsets

How many size k subsets of an size n set are there? Example: How many ways can we select 5 books from 100?

Let us form a permutation of the n elements, and pick the first k elements to form the subset. Every size k subset can be generated this way. There are $n!$ total such permutations.

The order of the first k elements in the permutation does not matter in forming the subset, and neither does the order of the remaining $n - k$ elements.

The first k elements can be ordered in $k!$ ways and the remaining $n - k$ elements can be ordered in $(n - k)!$ ways. Using the product rule, $k! \times (n - k)!$ permutations map to the same size k subset.

Hence, the function f : permutations $\rightarrow$ size k subsets is a $k! \times (n - k)!$-to-1 function. By the division rule, $|\text{permutations}| = k! \times (n - k)! \, |\text{size } k \text{ subsets}|$. Hence, the total number of size k subsets $= \frac{n!}{k! \times (n-k)!}$.

n **choose** $k = \binom{n}{k} := \frac{n!}{k! \times (n-k)!}$.

Counting subsets

Q: Prove that $\binom{n}{k} = \binom{n}{n-k}$ - both algebraically (using the formula for $\binom{n}{k}$) and combinatorially (without using the formula)

Ans: Algebraically, $\binom{n}{k}$ is symmetric with respect to $k$and $n-k$. Combinatorially, number of ways of choosing elements to form a set of size k = number of ways of choosing $n-k$ elements to discard.

Q: Which is bigger? $\binom{8}{4}$ vs $\binom{8}{5}$? Ans: $\binom{8}{4} = 70$. $\binom{8}{5} = 56$

Counting subsets - Example

How many m-bit binary sequences contain exactly k ones?

Consider set $A = \{a_1, \ldots, a_m\}$. If we select a S, a subset of size k, say $m = 10, k = 3$ and $S = \{a_3, a_7, a_{10}\}$. Subset S can mapped to the sequence 001 0001 001. Similarly, every m-bit sequence with exactly k ones can be mapped to a subset of size k.

Hence, there is a bijection:
f : m-bit sequence with exactly k ones $\rightarrow$ subsets of size k from size m set, and
$|m$-bit sequence with exactly k ones$| = |$subsets of size $k| = \binom{m}{k}$.

Recall that the number of ways of selecting 10 donuts with 5 varieties = number of 14-bit sequences with exactly 4 ones = $\binom{14}{4} = 1001$.

Q: What is the number of ways of choosing n things with k varieties?

Ans: Equal to the number of $n + k - 1$-bit sequences with exactly $k - 1$ ones = $\binom{n+k-1}{k-1}$.