CMPT 210: Probability and Computation

Lecture 14

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Recap

A **distribution** can be specified by its probability density function (PDF) (denoted by f).

Bernoulli Distribution: If random variable R follows the Bernoulli distribution i.e. $R \sim \text{Ber}(p)$, then $f_p(0) = 1 - p$, $f_p(1) = p$.

Uniform Distribution: If random variable $R : \mathcal{S} \to V$ follows the Uniform distribution i.e. $R \sim \text{Uniform}(V)$, then for all $v \in V$, $f(v) = 1/|V|$.

Binomial Distribution: If random variable R follows the Binomial distribution i.e. $R \sim \text{Bin}(n, p)$, then $f_{n,p}(k) = \binom{n}{k} p^k (1-p)^{n-k}$.

Geometric Distribution: If random variable R follows the Geometric distribution i.e. $R \sim \text{Geo}(p)$, then $f_p(k) = (1-p)^{k-1} p$.

Recap

Expectation/mean of a random variable R is denoted by $\mathbb{E}[R]$ and "summarizes" its distribution. Formally, $\mathbb{E}[R] := \sum_{\omega \in \mathcal{S}} \Pr[\omega] \, R[\omega]$

Example: When throwing a standard dice, if R is the random variable equal to the number on the dice. $\mathbb{E}[R] = \sum_{i \in \{1,2,\ldots,6\}} \frac{1}{6}[i] = \frac{7}{2}$.

Alternate definition of expectation: $\mathbb{E}[R] = \sum_{x \in \text{Range}(R)} x \, \Pr[R = x]$.

This definition does not depend on the sample space.

Example: If $\mathcal{I}_A$ is the indicator random variable for event A, then $\mathbb{E}[\mathcal{I}_A] = \Pr[\mathcal{I}_A = 1](1) + \Pr[\mathcal{I}_A = 0](0) = \Pr[A]$. For $\mathcal{I}_A$, the expectation is equal to the probability that event A happens.

Linearity of Expectation: For n random variables $R_1, R_2, \ldots, R_n$ and constants $a_1, a_2, \ldots, a_n$, $\mathbb{E}\left[\sum_{i=1}^{n} a_i R_i\right] = \sum_{i=1}^{n} a_i \, \mathbb{E}[R_i]$.

Recap

If $R \sim \text{Bernoulli}(p)$, $\mathbb{E}[R] = p$. *Example*: When tossing a coin, if R is the random variable equal to 1 if we get a heads.

If $R \sim \text{Uniform}(\{v_1, \ldots, v_n\})$, $\mathbb{E}[R] = \frac{v_1 + v_2 + \ldots + v_n}{n}$. *Example*: When throwing an n-sided dice with numbers $v_1, \ldots v_n$, if R is the random variable equal to the number.

If $R \sim \text{Bin}(n, p)$, $\mathbb{E}[R] = np$. *Example*: When tossing n independent coins, if R is the random variable equal to the number of heads.

If $R \sim \text{Geo}(p)$, $\mathbb{E}[R] = \frac{1}{p}$. *Example*: When tossing a coin repeatedly, if R is the random variable equal to the number of tosses required to get the first heads.

Expectation - Examples

Q: We throw a standard dice, and define a random variable R which is equal to 1 if we get an even number and 0 otherwise. What is the distribution of R? What is $\mathbb{E}[R]$? Ans: Ber(1/2), $\frac{1}{2}$

Q: We throw 10 independent dice and define R to be the random variable equal to the number of dice that have an even number. What is the distribution of R? What is $\mathbb{E}[R]$? Ans: Bin$(10, 1/2)$, 5

Q: We repeatedly and independently throw the dice until we get an even number. We define a random variable R equal to the number of throws we need to get an even number. What is the distribution of R? What is $\mathbb{E}[R]$? Ans: Geo(1/2), 2.

Expectation - Examples - Coupon Collector Problem

Q: In a game started by a coffee shop, each time we buy a coffee, we get a coupon. Each coupon has a color (amongst n different colors) and each time, the color of the coupon is selected uniformly at random from amongst the n colors. If we collect at least one coupon of each color, we can claim a free coffee. On average, how many coupons should we collect (coffees we should buy) to claim the prize?

Suppose we get the following sequence of coupons:

$$blue, green, green, red, blue, orange, blue, orange, gray$$

Let us partition this sequence into segments such that a segment ends when we collect a coupon of a new color we did not have before. For this example,

$$\underbrace{blue}_{S_1}\underbrace{green}_{S_2}\underbrace{green, red}_{S_3}\underbrace{blue, orange}_{S_4}\underbrace{blue, orange, gray}_{S_5}$$

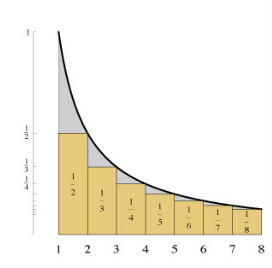
If the number of segments is equal to n, by definition, we will have collected coupons of the n different colors. Define X_k to be the random variable equal to the length of segment S_k and T to be the total number of coupons required to have at least one coupon per color.

Expectation - Examples - Coupon Collector Problem

$T = X_1 + X_2 + \ldots X_n$. We wish to compute $\mathbb{E}[T]$. By linearity of expectation, $\mathbb{E}[T] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \ldots + \mathbb{E}[X_n]$.

Let us calculate $\mathbb{E}[X_k]$. If we are on stage k, we have seen $k-1$ colors before. Hence, the probability of seeing a new (one that we have not seen before) colored coupon in S_k is $\frac{n-(k-1)}{n}$. $X_k \sim \text{Geo}\left(\frac{n-(k-1)}{n}\right)$, and we know that $\mathbb{E}[X_k] = \frac{n}{n-k+1}$.

$$\mathbb{E}[T] = \sum_{k=1}^{n} \frac{n}{n-k+1} = n\left[\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{1}\right]$$
$$\leq n\left[1 + \int_1^n \frac{dx}{x}\right] = n\left[1 + \ln(n)\right] \leq 2n\ln(n)$$

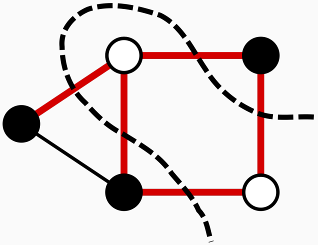


We also know that $\mathbb{E}[T] \geq n\ln(n+1)$. Hence, $\mathbb{E}[T] = O(n\ln(n))$, meaning that we need to buy $O(n\ln(n))$ coffees to collect coupons of n colors and get a free coffee.

Questions?

Max Cut

Given a graph $G = (\mathcal{V}, \mathcal{E})$, partition the graph's vertices into two complementary sets $\mathcal{S}$ and $\mathcal{T}$, such that the number of edges between the set $\mathcal{S}$ and the set $\mathcal{T}$ is as large as possible.



Max Cut has applications to VLSI circuit design.

Equivalently, find a set $\mathcal{U} \subseteq \mathcal{V}$ of vertices that solve the following

$$\max_{\mathcal{U} \subseteq \mathcal{V}} |\delta(\mathcal{U})| \text{ where } \delta(\mathcal{U}) := \{(u, v) \in \mathcal{E} | u \in \mathcal{U} \text{ and } v \notin \mathcal{U}\}$$

Here, $\delta(\mathcal{U})$ is referred to as the "cut" corresponding to the set $\mathcal{U}$.

Max Cut

- Max Cut is NP-hard (Karp, 1972), meaning that there is no polynomial (in $|\mathcal{E}|$) time algorithm that solves Max Cut exactly.
- We want to find an approximate solution $\mathcal{U}$ such that, if OPT is the size of the optimal cut, then, $|\delta(\mathcal{U})| \geq \alpha\, OPT$ where $\alpha \in (0, 1)$ is the multiplicative approximation factor.
- Randomized algorithm that guarantees an approximate solution with $\alpha = \frac{1}{2}$ with probability close to 1 (Erdos, 1967).
- Algorithm with $\alpha = 0.878$. (Goemans and Williamson, 1995).
- Under some technical conditions, no efficient algorithm has $\alpha > 0.878$ (Khot et al, 2004).

We will use Erdos' randomized algorithm and first prove the result in expectation. We wish to prove that for $\mathcal{U}$ returned by Erdos' algorithm,

$$\mathbb{E}[|\delta(\mathcal{U})|] \geq \frac{1}{2} OPT$$

. **Algorithm**: Select $\mathcal{U}$ to be a random subset of $\mathcal{V}$ i.e. for each vertex v, choose v to be in the set $\mathcal{U}$ independently with probability $\frac{1}{2}$ (do not even look at the edges!).

Max Cut

Claim: For Erdos' algorithm, $\mathbb{E}[|\delta(\mathcal{U})|] \geq \frac{1}{2}OPT$.

Proof: For each edge $(u, v) \in \mathcal{E}$, let $X_{u,v}$ be the indicator random variable equal to 1 iff the event $E_{u,v} = \{(u, v) \in \delta(\mathcal{U})\}$ happens.

$$\mathbb{E}[|\delta(\mathcal{U})|] = \mathbb{E}\left[\sum_{(u,v)\in\mathcal{E}} X_{u,v}\right] = \sum_{(u,v)\in\mathcal{E}} \mathbb{E}\left[X_{u,v}\right] = \sum_{(u,v)\in\mathcal{E}} \Pr[E_{u,v}]$$

$$\Pr[E_{u,v}] = \Pr[(u, v) \in \delta(\mathcal{U})] = \Pr\left[(u \in \mathcal{U} \cap v \notin \mathcal{U}) \cup (u \notin \mathcal{U} \cap v \in \mathcal{U})\right]$$

$$= \Pr\left[(u \in \mathcal{U} \cap v \notin \mathcal{U})\right] + \Pr\left[(u \notin \mathcal{U} \cap v \in \mathcal{U})\right]$$

$$\Pr[E_{u,v}] = \Pr[u \in \mathcal{U}]\,\Pr[v \notin \mathcal{U}] + \Pr[u \notin \mathcal{U}]\,\Pr[v \in \mathcal{U}] = \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{2} = \frac{1}{2}.$$

$$\implies \mathbb{E}[|\delta(\mathcal{U})|] = \sum_{(u,v)\in\mathcal{E}} \Pr[E_{u,v}] = \frac{|\mathcal{E}|}{2} \geq \frac{\text{OPT}}{2}.$$

Later in the course, we will prove that $|\delta(\mathcal{U})| \geq \frac{\text{OPT}}{2}$ with probability close to 1.

Questions?

Conditional Expectation

Similar to probabilities, expectations can be conditioned on some event.

For random variable R, the expected value of R conditioned on an event A is given by:

$$\mathbb{E}[R|A] = \sum_{x \in \text{Range}(R)} x \Pr[R = x|A]$$

Q: If we throw a standard dice and define R to be the random variable equal to the number that comes up, what is the expected value of R given that the number is at most 4? A is the event that the number is at most 4. $\Pr[R = 1|A] = \frac{\Pr[(R=1)\cap A]}{\Pr[A]} = \frac{\Pr[R=1]}{\Pr[A]} = \frac{1/6}{4/6} = 1/4$. Similarly, $\Pr[R = 2|A] = \Pr[R = 3|A] = \Pr[R = 4|A] = \frac{1}{4}$ and $\Pr[R = 5|A] = \Pr[R = 6|A] = 0$.

$$\mathbb{E}[R|A] = \sum_{x \in \{1,2,3,4\}} x \Pr[R = x|A] = \frac{1}{4}[1 + 2 + 3 + 4] = \frac{5}{2}.$$

Q: What is the expected value of R given that the number is at least 4? Ans: $\mathbb{E}[R|A] = \sum_{x \in \{4,5,6\}} x \Pr[R = x|A] = \frac{1}{3}[4 + 5 + 6] = 5$.

Law of Total Expectation

If R is a random variable $\mathcal{S} \to V$ and events $A_1, A_2, \ldots A_n$ form a partition of the sample space, then,

$$\mathbb{E}[R] = \sum_i \mathbb{E}[R|A_i] \Pr[A_i]$$

$$\begin{aligned}
\mathbb{E}[R] &= \sum_{x \in \text{Range}(R)} x \Pr[R = x] = \sum_{x \in \text{Range}(R)} x \sum_i \Pr[R = x|A_i] \Pr[A_i] \\
&= \sum_i \Pr[A_i] \sum_{x \in \text{Range}(R)} x \Pr[R = x|A_i] \\
&= \sum_i \Pr[A_i] \, \mathbb{E}[R|A_i].
\end{aligned}$$

Conditional Expectation - Examples

Q: Suppose that 49.6% of the people in the world are male and the rest female. If the expected height of a randomly chosen male is 5 feet 11 inches, while the expected height of a randomly chosen female is 5 feet 5 inches, what is the expected height of a randomly chosen person?

Define H to be the random variable equal to the height (in feet) of a randomly chosen person. Define M to be the event that the person is male and F the event that the person is female.
We wish to compute $\mathbb{E}[H]$ and we know that $\mathbb{E}[H|M] = 5 + \frac{11}{12}$ and $\mathbb{E}[H|F] = 5 + \frac{5}{12}$.
$\Pr[M] = 0.496$ and $\Pr[F] = 1 - 0.496 = 0.504$.
Hence, $\mathbb{E}[H] = \mathbb{E}[H|M]\Pr[M] + \mathbb{E}[H|F]\Pr[F] = \frac{71}{12}(0.496) + \frac{65}{12}(0.504)$.

Conditional Expectation - Examples

Recall that if $R \sim \text{Geo}(p)$, $\mathbb{E}[R] = 1/p$. To derive this, we computed the following sum $\mathbb{E}[R] = \sum_{k=1} k\,(1-p)^{k-1} p$. Let's use conditional expectation to do it in a simpler way.

For our coin tossing example, define R to be the random variable equal to the number of coin tosses required to get the first heads. Let A be the event that we get a heads in the first toss. Using the law of total expectation,

$$\mathbb{E}[R] = \mathbb{E}[R|A]\,\Pr[A] + \mathbb{E}[R|A^c]\,\Pr[A^c]$$

We know that, $\mathbb{E}[R|A] = 1$ ($R = 1$ if we get a heads in the first coin toss). $\Pr[A] = p$. Hence,

$$\mathbb{E}[R] = (1)(p) + \mathbb{E}[R|A^c](1-p)$$

$\mathbb{E}[R|A^c]$ is the expected number of tosses required to get the first heads *if* we do not get a heads on the first toss. Hence, $\mathbb{E}[R|A^c] = \mathbb{E}[R] + 1$.

$$\mathbb{E}[R] = (1)(p) + [1 + \mathbb{E}[R]](1-p) \implies \mathbb{E}[R] = 1 + \mathbb{E}[R] - p\mathbb{E}[R] \implies \mathbb{E}[R] = \frac{1}{p}.$$

Questions?

Randomized Quick Select

Given an array A of n distinct numbers, return the k^{th} smallest element in A for $k \in [1, n]$.

Algorithm Randomized Quick Select

```
function QuickSelect(A, k)
If Length(A) = 1, return A[1].
Select p ∈ A uniformly at random.
Construct sets Left := {x ∈ A | x < p} and Right := {x ∈ A | x > p}.
r = |Left| + 1 {Element p is the r^th smallest element in A.}
if k = r then
    return p
end if
if k < r then
    QuickSelect(Left, k)
else
    QuickSelect(Right, k − r)
end if
```

Randomized QuickSelect

If $A = \{2, 7, 0, 1, 3\}$ and we wish to find the 2^{nd} smallest element meaning that $k = 2$. According to the algorithm, $p \sim \text{Uniform}(A)$. Say $p = 3$.

Then after step 1, Left $= \{0, 1, 2\}$ and Right $= \{7\}$. $r := |\text{Left}| + 1 = 3 + 1 = 4$. Since $r > k$, we recurse on the left-hand side by calling the algorithm on $\{0, 1, 2\}$ with $k = 2$.

$p \sim \text{Uniform}(\{0, 1, 2\})$. Say $p = 1$. After step 2, Left $= \{0\}$ and Right $= \{2\}$. $r := |\text{Left}| + 1 = 1 + 1 = 2$. Since $r = k$, we terminate the recursion and return $p = 1$ as the second-smallest element in A.

Q: Run the algorithm if $p = 0$ in the first step? Ans: Left $= \{\}$ and Right $= \{1, 2, 3, 7\}$. Hence $r = 1 < k = 2$. Hence we will recurse on the right-hand side by calling the algorithm on $\{1, 2, 3, 7\}$ with $k = 1$.

Q: Run the algorithm if $p = 1$ in the first step? Ans: Left $= \{0\}$ and Right $= \{2, 3, 7\}$. Hence $r = 1 + 1 = 2$. Hence we will return the pivot element $p = 1$.

Randomized Quick Select – Analysis

Alternate way: Sort the elements in A and return the k^{th} element in the sorted list. Uses $O(n \log(n))$ comparisons.

Q: Can Randomized Quick Select do better – what is the maximum number of comparisons required by Randomized Quick Select (i) in the worst-case and (ii) in expectation (over the pivot selection)?

Claim: For any array A with n distinct elements, and for any $k \in [n]$, Randomized Quick Select performs fewer than $8n$ comparisons in expectation.

In order to prove this claim, we will need to prove the following Lemma.

Lemma: The child sub-problem's array (either Left or Right) after the partitioning (in Line 4 of the algorithm) has expected size smaller than $\frac{7n}{8}$.

Randomized Quick Select – Analysis

Let us define a "good" event $\mathcal{E}$ that the randomly chosen pivot splits the array roughly in half. Formally, if n is the length of the array, then $\mathcal{E}$ is the event that $r \in \left(\frac{n}{4}, \frac{3n}{4}\right]$ (for simplicity, let us assume that n is divisible by 4.) Since r is chosen randomly, $\Pr[\mathcal{E}] = \frac{3n/4 - n/4}{n} = \frac{1}{2}$.

Recall that $|\text{Left}| = r - 1$ and $|\text{Right}| = n - r$. Hence if event $\mathcal{E}$ happens, then $|\text{Left}| < \frac{3n}{4}$ and $|\text{Right}| < \frac{3n}{4}$. Hence, $|\text{Child}| < \frac{3n}{4}$. If event $\mathcal{E}$ does not happen, in the worst-case, $|\text{Child}| < n$. By using the law of total expectation,

$$\begin{aligned}\mathbb{E}[|\text{Child}|] &= \mathbb{E}[|\text{Child}| \mid \mathcal{E}] \Pr[\mathcal{E}] + \mathbb{E}[|\text{Child}| \mid \mathcal{E}^c] \Pr[\mathcal{E}^c] \\ &< \frac{3n}{4}\frac{1}{2} + (n)\frac{1}{2} = \frac{7n}{8}.\end{aligned}$$

Hence on average, the size of the child sub-problem is smaller than $\frac{7n}{8}$, proving the lemma.

Randomized Quick Select – Analysis

In order to upper-bound the total number of comparisons, we use the Lemma with an induction on n. Recall that we need to prove that Randomized Quick Select requires fewer than $8n$ comparisons in expectation.

Base case: If $n = 1$, then we require $0 < 8$ comparisons. Hence the base case is satisfied.

Inductive Step:

$$
\begin{aligned}
&\mathbb{E}[\text{Total number of comparisons for size } n \text{ array}] \\
&= \mathbb{E}[(n-1) + \text{Total number of comparisons in child sub-problem}] \\
&= (n-1) + \mathbb{E}[\text{Total number of comparisons in child sub-problem}] \quad \text{(Linearity of expectation)} \\
&< (n-1) + 8\,\mathbb{E}[|\text{Child}|] \quad \text{(Induction hypothesis)} \\
&< (n-1) + 8\frac{7n}{8} < 8n. \quad \text{(Lemma)}
\end{aligned}
$$

Hence we have proved our claim that for any $k \in [n]$, on average, Randomized Quick Select requires fewer than $8n$ comparisons.

Questions?

Independence of random variables

We define two random variables R_1 and R_2 to be independent if for *all* $x_1 \in \text{Range}(R_1)$ and $x_2 \in \text{Range}(R_2)$, events $[R_1 = x_1]$ and $[R_2 = x_2]$ are independent. More formally, we require,

$$\Pr[(R_1 = x_1) \cap (R_2 = x_2)] = \Pr[(R_1 = x_1)] \Pr[(R_2 = x_2)]$$

Q: Suppose we toss three independent, unbiased coins. Let C be r.v. equal to the number of heads that appear and M be the r.v. that is equal to 1 if all the coins match. Are random variables C and M independent?

$\text{Range}(C) = \{0, 1, 2, 3\}$ and $\text{Range}(M) = \{0, 1\}$. $\Pr[C = 3] = \frac{1}{8}$ and $\Pr[M = 1] = \frac{1}{4}$. $\Pr[(C = 3) \cap (M = 1)] = \frac{1}{8} \neq \frac{1}{32} = \Pr[C = 3] \Pr[M = 1]$. Hence, C and M are not independent.