Assignment 2
CMPT 210

Due: In class on Friday, 17 June

(1) [20 marks] Three cards are drawn one after the other from an ordinary 52-card deck without replacement (once a card is drawn, it is not placed back in the deck). Compute the probability that

(i) All of the three cards is a heart. [5 marks]

(ii) None of the three cards is a heart. [5 marks]

(iii) Exactly one of the three cards is a heart. [10 marks]

(2) [10 marks] Suppose we have k computer servers and m jobs that needs to be assigned to these servers. We are using a random allocation strategy such that each job is randomly assigned to a server. A server fails if there is more than one job assigned to it. Calculate the probability that no server fails.

(3) [25 marks] A class consisting of 4 graduate and 12 undergraduate students is randomly divided into four groups of 4.

- What is the number of possible groups that can be formed? [10 marks]
- Suppose we impose the restriction that each group needs to have exactly one graduate student, what is the number of possible groups that can be formed? [10 marks]
- What is the probability that each group includes exactly one graduate student? [5 marks]

(4) [20 marks] Let us consider a game played between two players rolling a "standard" dice. The players roll the dice alternatively, and the first player that rolls a 6 wins the game.

- What is the probability that Player 1 (who starts the game) wins with their first roll of the dice? [2 marks]
- What is the probability that Player 1 (who starts the game) wins with their second roll of the dice? [5 marks]
- What is the probability that Player 1 (who starts the game) eventually wins the game? [13 marks]

(5) **[25 marks]** You shuffle a deck of cards and deal your friend 5 cards.

- What is the probability that they have exactly one ace in their 5 cards? [5 marks]
- What is the probability that they have the ace of spades in their 5 cards? [5 marks]
- Suppose your friend says, "I have the ace of spades". What is the probability that they have exactly two aces? [5 marks]
- Suppose your friend says, "I have at least one ace". What is the probability that they have at least two aces? [10 marks]

(6) **[30 marks]** Professor Plum, Mr. Green, and Miss Scarlet are all plotting to shoot Colonel Mustard. If one of these three has both an opportunity and the revolver, then that person shoots Colonel Mustard. Otherwise, Colonel Mustard escapes. Exactly one of the three has an opportunity with the following probabilities:

$$\Pr[\text{Plum has the opportunity}] = \frac{1}{6}$$
$$\Pr[\text{Green has the opportunity}] = \frac{2}{6}$$
$$\Pr[\text{Scarlet has the opportunity}] = \frac{3}{6}$$

Exactly one has the revolver with the following probabilities, regardless of who has an opportunity:

$$Pr[\text{Plum has the revolver}] = \frac{4}{8}$$
$$\Pr[\text{Green has the revolver}] = \frac{3}{8}$$
$$\Pr[\text{Scarlet has the revolver}] = \frac{1}{8}$$

- Draw a tree diagram for this problem. Indicate edge and outcome probabilities. [15 marks]
- What is the probability that Colonel Mustard is shot? [5 marks]
- What is the probability that Colonel Mustard is shot, given that Miss Scarlet does not have the revolver? [5 marks]
- What is the probability that Mr. Green had an opportunity, given that Colonel Mustard was shot? [5 marks]

(7) **[15 marks]** There is a rare disease called Nerditosis which afflicts about 1 CS student in 1000. One symptom of Nerditosis is a compulsion to refer to everything using numbers or Greek symbols. Two professors claim that they can diagnose Nerditosis.

- If the student has Nerditosis, Professor X can detect it with probability 0.99.
- If the student does not Nerditosis, Professor X will erroneously detect it with probability 0.03.
- Professor Y does not have the sophisticated equipment that Professor X has, but he knows that Nerditosis affects one in thousand students on average. Hence, when asked if a student has Nerditosis, Professor Y says "yes" with probability $\frac{1}{1000}$.

- What is the probability that Professor X misdiagnoses Nerditosis i.e. says "yes" when the student does not have it or "no" when the student does indeed have it? [7 marks]
- What is the probability that Professor Y misdiagnoses Nerditosis? [7 marks]
- Which professor is more accurate in detecting Nerditosis (has a smaller probability of making a misdiagnosis)? [1 marks]

(8) [15 marks] The SFU football team has probability 0.3 of winning against Tier 1 teams, probability 0.4 of winning against Tier 2 teams and probability 0.5 of winning against Tier 3 teams. Half the teams in the league are Tier 1 while a quarter of them are Tier 2 and a quarter of them are Tier 3. The odds of an event A is given by:

$$\text{odds[A]} = \frac{\Pr[A]}{1 - \Pr[A]}$$

- What are the odds that the SFU football team wins against a randomly chosen team? [10 marks]
- Given SFU wins the game, what is the probability that the opponent was a Tier 2 team? [5 marks]

(9) [25 marks] Prove the following statements.

- For an event A and two mutually exclusive events E_1 and E_2,

$$\Pr[E_1 \cup E_2 | A] = \Pr[E_1|A] + \Pr[E_2|A]$$

  implying that the union-rule for mutually exclusive events also works if we condition on the event A. [10 marks]

- For an event A and two mutually exclusive events E_1 and E_2, prove the following statement. [10 marks]

$$\Pr[A|E_1 \cup E_2] = \frac{\Pr[A|E_1]\ \Pr[E_1] + \Pr[A|E_2]\ \Pr[E_2]}{\Pr[E_1] + \Pr[E_2]}$$

- Prove that the above rule generalizes the law of total probability, i.e. if $E_1 \cup E_2 = \mathcal{S}$, then the above equation recovers the law of total probability, i.e. $\Pr[A] = \Pr[A|E_1]\ \Pr[E_1] + \Pr[A|E_1^c]\ \Pr[E_1^c]$. [5 marks]

(10) [20 marks] Suppose that an insurance company classifies people into one of three classes – good risks, average risks, and bad risks. Their records indicate that the probabilities that good, average, and bad risk persons will be involved in an accident over a 1-year span are, respectively, 0.05, 0.15, and 0.30. If 20% percent of the population are "good risks", 50% percent are "average" risks and 30% percent are "bad risks",

- What is the probability that a person has an accident in a fixed year? [10 marks]
- If a policy holder had no accidents in the 1-year span, what is the probability that they are (i) good (ii) average (iii) bad risk? [4 + 3 + 3 marks]

(11) [15 marks] We roll a "standard" dice twice.

- A_1 is the event that the first roll is 1 and B_1 is the event that the sum of the numbers on the two dice is equal to 5. Calculate $\Pr[A_1]$, $\Pr[B_1]$ and $\Pr[A_1 \cap B_1]$ and ascertain whether the events A_1 and B_1 independent? [5 marks]

- A_2 is the event that the maximum of the two rolls is 2 and B_2 is the event that minimum of the two rolls is 2. Calculate $\Pr[A_2]$, $\Pr[B_2]$ and $\Pr[A_2 \cap B_2]$ and ascertain whether the events A_2 and B_2 independent? [10 marks]

(12) [30 marks] Implementing Frievald's algorithm.

- Write a function in the language of your choice (preferably C or Python) that implements Basic Frievald's algorithm we learned in Lecture 10. [10 marks]

- Use this function with the probability amplification scheme and write a function for the complete Frievalds algorithm. [10 marks]

Print out your code and submit it with the assignment.

Use the following matrices in order to test the code to verify that $C = AB \bmod 2$ and $D \neq AB \bmod 2$:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

We can compute the probability of failure of a randomized algorithm by repeating the experiment for 1000 independent runs (each run of the algorithm involves generating a new independent random vector x). The pseudo-code to test the Basic Frievald's algorithm is provided below (use a similar setup for the complete Frievald's algorithm).

#num_runs = 1000 ; #count_correct = 0 ; #count_incorrect = 0
#for _ in range(num_runs):{
For inputs A, B, C if Basic Frievald's says "yes", count_correct++
For inputs A, B, D if Basic Frievald's says "yes", count_incorrect++
#}
Compute $1 - \frac{\text{count\_correct}}{\text{num\_runs}}$; $\frac{\text{count\_incorrect}}{\text{num\_runs}}$.

- Report $1 - \frac{\text{count\_correct}}{\text{num\_runs}}$; $\frac{\text{count\_incorrect}}{\text{num\_runs}}$ for the Basic Frievalds and the complete Frievalds algorithm (with 20 trials). [10 marks]