Assignment 1
CMPT 210

Due: In class on Friday, 27 May

1 Sets [20 marks]

(a) For $\mathbb{N} = \{0, 1, 2, \ldots\}$, enumerate the elements of the set $A = \{x \in \mathbb{N} | x^2 + x - 6 \leq 0\}$. [5 marks]

(b) In Lecture 1, we proved the distributive law for intersection,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Prove the distributive law for union,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

using both (i) Venn diagrams and (ii) the distributive law: x OR (y AND z) = (x OR y) AND (x OR z). [5 + 5 marks]

(c) The power set pow(A) of a set A is the set of all subsets formed from the elements of A. For example, if $A = \{1, 2\}$, the pow(A) $= \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Prove that for any set A, $|\text{pow}(A)| = 2^{|A|}$ [5 marks].

2 Functions [20 marks]

(a) For functions $f : A \to B$ and $g : B \to C$, the function composition denoted as $g \circ f$ is a function from $A \to C$ such that $(g \circ f)(x) = g(f(x))$. For example, function $f : \mathbb{R}_+ \to \mathbb{N}$ such that $f(x) = \lceil x \rceil$ corresponds to the *ceiling* operation that rounds a real number to the nearest integer larger than the number. E.g. $f(2.3) = 3$. Function $g : \mathbb{N} \to \mathbb{N}$ is the mod 41 function we saw in Lecture 3 such that $g(x) = x \bmod 41$. E.g. $g(52) = 52 \bmod 41 = 11$.

- What is the domain, codomain, range of f? [2 marks]
- What is the domain, codomain, range of g? [2 marks]
- What is the domain, codomain, range of $g \circ f$? [4 marks]
- Calculate the value of $(g \circ f)(41.1) + f(41.1) + g(0)$. [2 marks]

(b) The inverse of a bijective function $s : A \to B$ is the function $s^{-1} : B \to A$ such that for $x \in A, y \in B$ if $y = s(x)$, then $x = s^{-1}(y)$. For example, consider $f(x) = \lceil x \rceil$, $g(x) = x \bmod 41$ from the previous question and $h : \mathbb{R}_+ \to \mathbb{R}$ such that $h(x) = \log_{10}(x)$.

- For a bijective function $s : A \to A$, prove that $(s^{-1} \circ s)(x) = x$. [4 marks]
- Is $f(x)$ invertible i.e. is f^{-1} a function? If so, compute $f^{-1}(3)$. [2 marks]
- Is $g(x)$ invertible i.e. is g^{-1} a function? If so, compute $g^{-1}(11)$. [2 marks]
- Is $h(x)$ invertible i.e. is h^{-1} a function? If so, compute $h^{-1}(10)$. [2 marks]

3 Counting [105 marks]

(a) A license plate consists of either:

- 3 upper-case letters followed by 3 digits (standard plate)
- 5 upper-case letters (vanity plate)
- 2 characters – each of which is either an upper-case letter or number (big shot plate)

If P is the set of all possible plates, and $L = \{A, B, \dots, Z\}$ is the set of upper-case letters, and $D = \{0, 1, \dots, 9\}$ is the set of digits,

- Express P in terms of L, D using the set union $\cup$ and product $\times$ operations. [10 marks]
- Using the sum rule and the product rule, compute $|P|$. [5 marks]

(b) In Lecture 3, we used a combinatorial technique to prove Pascal's identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Using the definitions of $\binom{n}{k}$ and $n!$, prove Pascal's inequality algebraically. [10 marks]

(c) Given m distinct numbers, how many $n \times n$ matrices are possible such that the matrix has all distinct entries? Assume that $m > n^2$. [10 marks]

(d) In Lecture 3, we saw the Binomial Theorem – for all $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

and generalized it to the Multinomial Theorem – for all $m, n \in \mathbb{N}$ and $z_1, z_2, \dots z_m \in \mathbb{R}$,

$$(z_1 + z_2 + \dots + z_m)^n = \sum_{\substack{k_1, k_2, \dots, k_m \\ k_1 + k_2 + \dots k_m = n}} \binom{n}{k_1, k_2, \dots, k_m} z_1^{k_1} z_2^{k_2} \dots z_m^{k_m}$$

where $\binom{n}{k_1, k_2, \dots, k_m} = \frac{n!}{k_1! k_2! \dots k_m!}$.

- What is the number of terms in the expansion $(a+b)^n$? [5 marks]
- What is the number of terms in the expansion $(z_1 + z_2 + \dots + z_m)^n$? [10 marks]

(e) For an undirected graph with n vertices $\{v_1, v_2, \ldots, v_n\}$,

- What is the maximum possible number of edges if i) self-loops (edges of the form $v_1 \to v_1$) are not permitted, ii) if self-loops are permitted? [5 marks]
- Given the answer to the previous question, what is the total number of possible graphs that can be constructed if i) self-loops (edges of the form $v_1 \to v_1$) are not permitted, ii) if self-loops are permitted? [5 marks]

(f) What is the total number of permutations of the letters (i) BANANA (ii) APPLE (iii) GRAPES (iv) A'PPLE (v) BANANA' where A' is a new letter of the alphabet and is not identical to A. [5 x 2 marks]

(g) A number in $\{6 \ldots, 48\}$ is composite iff it is divisible by either 2, 3 or 5. If D_i is the set of numbers divisible by i for $i \in 2, 3, 5$,

- Compute $|D_i|$ for $i \in \{2, 3, 5\}$. [3 marks]
- Compute $|D_i \cap D_j|$ for $i, j \in \{2, 3, 5\}$ and $i < j$. [3 x 2 marks]
- Compute $|D_2 \cap D_3 \cap D_5|$. [3 marks]
- Use the inclusion-exclusion principle to compute the number of prime numbers in $\{6, \ldots, 48\}$. [3 marks]

(h) Consider tossing a fair coin 100 times and let us record the sequence. For example, if we toss the coin 3 times, we might get the sequence HHT corresponding to heads in the first 2 tosses and tails in the third toss. For 100 tosses of the coin, What is the number of sequences in which we can get with, [4 x 5 marks]

- 25 heads, 75 tails
- 50 heads, 50 tails
- 75 heads, 25 tails
- 100 heads, 0 tails

4 Pigeonhole principle [25 marks]

(a) SFU ID numbers are 9 digit numbers that start with 3. How many students do we need in this class such that there are at least two students with the same sum of their SFU ID digits? [10 marks]

(b) Prove that for any 5 points in (the interior of) a unit square (one that has side length = 1), there exist 2 points at distance less than $\frac{1}{\sqrt{2}}$. [15 marks]